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APPLIED MECHANICS

BY

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PREFACE TO FIRST EDITION

This book covers the ground of the third or final year of a three-year part-time course of instruction leading to an Ordinary Grade National Certificate in Mechanical Engineering. During the first and second years a student will have become acquainted with most of the few but important principles of mechanics. In the final year the principles are developed and applied to problems which do not require much detailed technical knowledge of engineering practice. But the emphasis is on application. In this course, the way is prepared for a more advanced study of the theory of machines and the mechanics of materials and structures in the courses leading to the Higher National Certificate.

The scheme for the award of National Certificates provides for some freedom of choice and differences in syllabuses. Thus an item may appear in the second-year course in one college and in the third year in another. Consequently, this third-year text-book includes some small portions, *e.g.* in Hydraulics, which are substantially the same as in a second-year book such as "Mechanical Engineering Science" (Morley and Hughes). Such portions can easily be omitted by students who have studied them in an earlier course, but they are included for the use of students of colleges which defer these sections until the third year. Some work is also included, *e.g.* in the chapter on Structures, which may not appear in any year of some Mechanical National Certificate courses.

The reader who has worked through this book should be able to take the Applied Mechanics' examination for

Associate Membership of the major professional engineering institutions.

I wish to thank my friend Dr. Edward Hughes for valuable help in correction of proofs and suggesting emendations.

ARTHUR MORLEY.

Bath,
March, 1943.

PREFACE TO SECOND EDITION

An Appendix of 80 questions with numerical answers has been compiled from the Third Year, Senior (S_3) examination papers of four examining bodies viz East Midland Education Union, Northern Counties Technical Examinations Council, Union of Educational Institutions and Union of Lancashire and Cheshire Institutes. To these bodies I express my thanks for permission to reproduce their questions.

A. M.

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CHAPTER I

PRELIMINARY

1. Introductory. In order to apply and extend previous courses in mechanics, such as " Elementary Engineering Science " and " Mechanical Engineering Science " (Morley and Hughes), it is desirable as a starting point to have for reference some fairly definite statements of the terms used in the subject. This first chapter is devoted to definitions and brief restatements of important principles with which most readers will have had some prior acquaintance. These are partly reminders and partly a basis for application and extension.

As the title implies this book is concerned with application rather than logical development of principles, and it is permissible to choose in any problem the most convenient of several known relationships between quantities in order to obtain a solution. We can in effect take any established principles as tools for use in engineering problems without being obliged to make the tools. But practice in the use of such tools will reveal the possibilities of their wider use.

At about this stage in his studies a student will normally acquire some knowledge of the calculus. Even the notation will be useful to express briefly and clearly important relationships and principles. But as not all readers will have this knowledge readily at their disposal when beginning this stage of their studies of mechanics, the use of the calculus notation in this chapter is presented only as an alternative form (Art. 15). It should, however, be realized that it offers great advantages.

2. Velocity. The velocity of a point is the rate of its displacement and, like displacement, it has definite direction as well as magnitude. *Speed*, or the magnitude of a velocity, without regard to direction, is measured in units of length per unit of time, such as feet per second or miles per hour.

If displacements are all the same in equal intervals of time, the velocity is constant or uniform. In this case

$$\text{speed} = \text{distance moved/time taken} \quad . \quad . \quad (1)$$

which may be written,

$$v = s/t; \text{ or } s = vt; \text{ or } t = s/v \quad . \quad . \quad . \quad (2)$$

where s is the magnitude of the displacement expressed in linear units and t is the time taken for the displacement. The commonest units are

$$v = s \text{ ft.}/t \text{ sec.} = s/t \text{ ft. per sec.} \quad . \quad . \quad . \quad (3)$$

If the displacements in equal intervals of time are not equal, the velocity is variable and the above statements (1), (2), and (3) then refer to *average* or *mean* speeds, and the actual speed is the limit to which the average speed approaches when estimated over a very short interval of time. Or the numerical value of the speed at any instant is equal to the number of units of length which would be covered in one unit of time if the speed ceased to change.

When a rigid body is in motion, the different parts of that body generally have velocities differing from one another in magnitude or direction or in both, and the foregoing definitions will apply only to particular points and not to the body as a whole.

The velocity of a point, like its displacements, can be represented completely by a straight line of definite length and direction, *i.e.* by a vector, and it is subject to vectorial addition of its component parts and is capable of resolution into components in arbitrary directions generally chosen at right angles to one another.

Angular velocity (ω) of a point about an axis is the rate of angular displacement about the axis, and is generally measured in radians per second or revolutions per minute. When motion is wholly in one plane, angular displacement and angular velocity are often referred to as being about a point in the plane instead of about an axis.

$$\left.\begin{aligned} \omega &= \frac{\text{radians}}{\text{seconds}} = \frac{\text{arc}}{\text{radius} \times \text{seconds}} = \frac{vt}{r \times t} = \frac{v}{r} \\ \text{or} \quad \omega &= \text{linear velocity/radius.} \end{aligned}\right\} \quad . \quad (4)$$

3. Acceleration. The acceleration of a point is the rate of increase of its velocity; when negative, *i.e.* when velocity is decreasing, the rate of change is called retardation or deceleration. The various points in a moving body will generally have different accelerations. Like velocity, acceleration can be either constant or varying. If the acceleration f is constant,

$$f=\frac{\text{change of velocity}}{\text{time of change}}=\frac{v_2-v_1}{t} \quad . \quad . \quad . \quad (1)$$

or

$$v_2=v_1+ft. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where v_1=initial velocity and v_2=final velocity. But the subtraction, v_2-v_1, can only be made arithmetically if the direction is constant, that is, for motion in a straight line. In general, the change in velocity is a vector change and has direction, and the change must be found by vectorial subtraction (Art. 18). Thus linear acceleration is a vector quantity having definite direction as well as magnitude.

For motion in a straight line with constant acceleration f, the distance moved in time t is

$$s=\tfrac{1}{2}(v_2+v_1)t=\tfrac{1}{2}(2v_1+ft)t=v_1t+\tfrac{1}{2}ft^2 \quad . \quad . \quad (3)$$

Similarly for angular motion, the angular displacement, θ radians is

$$\theta=\omega_1t+\tfrac{1}{2}\alpha t^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where ω_1 is the initial angular velocity in radians per second and α is the angular acceleration in radians per second per second.

From (1) and (3)

$$v_2^2-v_1^2=2fs \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and similarly

$$_2^2-\omega_1^2=2\alpha\theta \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where ω_2 is the final angular velocity.

4. Mass and Inertia. The reluctance of a body to take up motion or to change its motion when acted upon by force is called its inertia. When expressed quantitatively, the measure of the reluctance of a body to change its

linear velocity is called its *mass*. The force exerted upon a body by the earth's attraction is called its *weight*.

Units of Mass. The best known British unit of mass is the mass of the standard pound (of platinum), but a common unit referred to below is about 32·2 pounds.

5. Force. If we regard force as the cause of change in the velocity of a body (whether in direction or in magnitude or in both), *i.e.* the acceleration of a body, we may take as a measure of the force the acceleration it produces in the body.

Let W lb. be the weight of a body and let g feet per sec. per sec. be the acceleration with which it would fall freely towards the earth if dropped; that is with a force equal to W lb. weight exerted upon it. Then if the acceleration of the same body when subjected to a force equal to P lb. weight is f ft. per sec. per sec., by experiment it is found that

$$\frac{f}{g}=\frac{\mathrm{P}}{\mathrm{W}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\mathrm{P}=\frac{\mathrm{W}}{g} . f=mf . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $m=\mathrm{W}/g$, a factor which together with the acceleration measures the force, or W/g is a measure of the inertia and can be adopted as a measure of mass. This measure is constant for any body provided W is measured in terms of the ratio of the force of the earth's pull on the body at any particular place to the pulling force of the earth on the standard lb. of platinum at London and g is acceleration due to gravitation at the particular place. For then W and g vary from place to place in the same proportion. The unit of force is then the weight of the standard pound mass in a standard place, viz. in London for the British Engineer's unit of force. Thus, (2) becomes

$$\text{P, in lb. force}=\frac{\mathrm{W}\ \text{lb. force}}{g\ \text{ft./sec.}^2}\times f\ \text{ft./sec.}^2 \quad . \quad (3)$$

For academic purposes, if the unit of mass is taken as that of the standard lb. of platinum and the unit of

acceleration as one ft. per sec. per sec., the unit of force is the poundal, so that

$$\text{P (poundals)} = m(\text{lb}) \times f(\text{ft./sec.}^2) \quad . \quad . \quad . \quad (3)$$

For physical science the unit of mass is often the gramme and the unit of acceleration is one cm./sec./sec., so that

$$\text{P (dynes)} = m(\text{gramme}) \times f(\text{cm./sec.}^2) \quad . \quad . \quad (4)$$

6. Momentum. The momentum of a body of mass m, all parts of which are moving with the same velocity v is given by

$$\text{momentum} = \text{mass} \times \text{velocity} = mv \text{ or } \frac{W}{g}v \quad . \quad (1)$$

The relation of (1) and (2) of Art. 5 may be written

$$\text{Force} = \text{P} = mf = m \times \text{rate of change of velocity} \quad . \quad (2)$$
$$= \text{rate of change of } mv \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$
$$= \text{rate of change of momentum} \quad . \quad . \quad . \quad (4)$$

Like velocity, momentum is a vector quantity.

7. Impulse. The impulse of a constant force P exerted on a body for a time t is $\text{P} \times t$, and from (2) of Art. 6, multiplying each side by t,

$$\text{P}t = m \times \text{total change of velocity in time } t \quad . \quad (1)$$
$$= \text{total change of } mv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$= \text{total change of momentum} \quad . \quad . \quad . \quad . \quad (3)$$

If the force is variable, the value of P in (1) must be its mean value in the time t, or the impulse must be found as the sum of a number of parts, the force P being considered constant over each of a number of intervals. Such a summation may be carried out graphically, or by arithmetical tabulation or by the calculus.

8. Moment, Torque and Centre of Gravity. The moment of a force about any axis perpendicular to its line of action is the magnitude of the force multiplied by the perpendicular distance of the force from the axis. This distance is called the arm or leverage of the force about the axis.

In plane or two-dimensional problems it is often convenient to refer to moments about a point; such a point is the intersection of an axis with the (perpendicular) plane of the forces.

The units of the moment of a force are pound-inches or pound-feet for forces in pounds and lengths in inches or feet respectively, and correspondingly for other units of force and length. The dimensions of moment are the same as those of work or mechanical energy, but it is convenient to distinguish them by putting the force first, as in pound-feet, for moments.

The word *torque* is often used instead of moment of force, particularly when reference is made to a turning moment about a real axle rather than an imaginary line or axis used for calculation or argument in statics.

Centre of Gravity and Centroid. If a body be divided into small parts of mass m_1, m_2, m_3, etc., distant x_1, x_2, x_3, etc., respectively, from a plane, then

$$g(m_1x_1 + m_2x_2 + m_3x_3 + \text{etc.}) \text{ or } g\,\Sigma(mx),$$

as the parts are considered smaller without limit, may be called the plane moment of the weight of the body with respect to the plane. The mean distance of the mass of the body from the plane is

$$\bar{x} = \Sigma(mx)/\Sigma(m) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is the distance of the centre of gravity or centre of mass from the plane. If all the elements of mass (m) are in one plane, then equation (1) may represent the distance of the centre of gravity from an axis in that plane. And similarly the distance of the centre of gravity of a lamina or the centroid of an area from an axis in its plane is

$$\bar{x} = \Sigma(ax)/\Sigma(a) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where the parts a are small divisions of the total area $\Sigma(a)$ and values of x are their distances from the axis.

9. Moment of Inertia. If the material of a body be divided into very small parts, the masses of which are m_1, m_2, m_3, etc., situated at distances r_1, r_2, r_3, etc., respectively, from some axis, then the limit to which the sum

$$m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \text{etc.}$$

tends, as the parts are considered smaller and smaller without limitation, is called the moment of inertia of the body about that axis and is generally denoted by I. Thus

$$I = \Sigma(mr^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The same term and symbol is used for the moment of inertia of an area about an axis, that is, the moment of inertia of an indefinitely thin and massless lamina of area A, which might be divided into small component areas a_1, a_2, a_3, etc. In this case, in the limit of subdivision

$$\text{moment of inertia of area} = I = \Sigma(ar^2) \quad . \quad (2)$$

This is also called the *second moment* of the area. For moments of inertia about different parallel axes, see Theorem (1) of Appendix.

Radius of Gyration. The radius at which, if the whole mass $\Sigma(m)$ or M were concentrated, the moment of inertia would be I, as in (1), is called the radius of gyration k, so that

$$Mk^2 = I = \frac{W}{g}k^2 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where W is the weight of the mass M, and

$$k^2 = I/M \text{ or } \Sigma(mr^2)/\Sigma(m) \quad . \quad . \quad . \quad (4)$$

And in the case of an area

$$k^2 = I/A \text{ or } \Sigma(ar^2)/\Sigma(a) \quad . \quad . \quad . \quad . \quad (5)$$

For radii of gyration about different parallel axes, see Theorem (1) in Appendix.

Units of Moment of Inertia. If the pound is adopted as the unit of mass then the unit of moment of inertia of a mass will be that of one pound at a radius of one foot. But if W/g is taken as the measure of mass (*i.e.* a unit of 32·2 lb. is adopted), then the unit of moment of inertia will be that of 32·2 lb. at a radius of 1 foot. Both units are used and the unit used for I should be specified.

10. Angular Momentum. If a rigid body is rotating about an axis with angular velocity ω (see the notation

of the preceding article), the velocities of the parts will be $r_1\omega$, $r_2\omega$, $r_3\omega$, etc.; and their momenta m_1v_1, m_2v_2, m_3v_3, etc., will be $m_1r_1\omega$, $m_2r_2\omega$, $m_3r_3\omega$, etc.; and the moments of the momenta about the axis of rotation will be

$$m_1r_1^2\omega,\ m_2r_2^2\omega,\ m_3r_3^2\omega,\ \text{etc.},$$

and the sum of these, since ω is the same for all parts, will be

$$\Sigma(mr^2\omega)=\omega\,\Sigma(mr^2)=\mathrm{I}\omega \text{ or } Mk^2\omega \quad . \quad . \quad (1)$$

or
$$\Sigma(mr^2\omega)=(\mathrm{W}/g)k^2\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where W is the weight of the body. This quantity $\mathrm{I}\omega$, the total moment of momentum of the body, is also called the angular momentum of the body about the axis of rotation.

11. Rotational Inertia. If the body referred to in the two preceding articles has an angular acceleration α about the axis of rotation, the linear accelerations of the masses m_1, m_2, m_3, etc., will be αr_1, αr_2, αr_3, etc.; and the forces necessary to give these (see (2), Art. 5) will be

$$m_1r_1\alpha,\ m_2r_2\alpha,\ m_3r_3\alpha,\ \text{etc.},$$

and the total moment or torque for the body will be

$$\mathrm{T}=m_1r_1^2\alpha+m_2r_2^2\alpha+m_3r_3^2\alpha+\text{etc.}=\alpha\,\Sigma(mr^2)=\mathrm{I}\alpha\,. \quad (1)$$

which is a relation closely corresponding to (2) of Art. 5 referring to linear motion. Comparison of the two expressions shows that I or $\mathrm{W}k^2/g$ is analogous to M or W/g and α is analogous to f.

If T is in units of pound-feet, then I will be in units equal to 32·2 lb. at 1 ft. radius; but if T is in poundal-feet, then I will be in units of 1 lb. at 1 ft. radius.

12. Work. A force exerted through a definite distance against a resistance is said to do work. The amount of work done is the product of the displacement or distance and the force, estimated in the direction of the displacement. Thus if a constant force of P pounds is exerted through a length of s feet and is inclined at angle θ to this displacement,

$$\text{Work}=\mathrm{P}s\cos\theta \text{ foot-pounds} \quad . \quad . \quad . \quad (1)$$

$$\text{And if } \theta=0,\ \text{Work}=\mathrm{P}\,.\,s\,. \quad . \quad . \quad . \quad (2)$$

If P is not constant but varies with the displacement, then

$$\text{Work} = \text{S} \times (\text{average value of P} \cos\theta) \text{ foot-pounds} . \quad (3)$$

the average being a *space-average*, *i.e.* the force which would be represented by the mean height of a diagram on which the vertical ordinates represent the varying values of P and the horizontal ordinates represent the corresponding displacements.

Diagrams of work in which the two ordinates represent force and linear displacement respectively will be familiar to the reader. In such a diagram the graphical equivalent of (3) is that the work done is represented by the area under a curve, the ordinates of which represent P cos θ on a base line representing s or by the product of the average height of the curve and the length of base. Reciprocating heat-engine indicator diagrams are well known examples of such graphical representation.

Angular motion. The work of a constant couple or turning moment or torque T pound-feet in turning through an angular displacement θ radians is

$$\text{Work} = \text{T} \times \theta \text{ foot-pounds} . \quad . \quad . \quad . \quad (4)$$

and for a torque T which varies with θ,

$$\text{Work} = \theta \times \text{average value of T} . \quad . \quad . \quad (5)$$

the average value being that represented by the mean height of a diagram on which the vertical ordinates represent the varying values of T and the horizontal ordinates represent the corresponding angular displacements. In particular, if T varies directly in proportion to the displacement θ, the average value of T is $\frac{1}{2}\text{T}_1$, where T_1 is the maximum value and

$$\text{Work done} = \tfrac{1}{2}\text{T}_1\theta . \quad . \quad . \quad . \quad . \quad (6)$$

13. Mechanical Energy. (*a*) *Potential Energy.* A body which because of its position can do work by a change of position is said to possess potential energy. A change of level under the action of gravitational force is a very common example; the total amount of potential energy in a body at any particular level depends upon some datum

level to which the body might be lowered, but we are mainly concerned not with the total reckoned from an arbitrary level but with changes in energy with changes in level.

If a body of weight W lb. can be lowered through a height h ft., it can do work, viz.

$$\text{Work done} = W.h \text{ foot-pounds} \quad . \quad . \quad (1)$$

and is said to have stored or potential energy of Wh foot-pounds.

In a transmission or a transformation of mechanical energy the work done is the measure of the energy transmitted or transformed.

(*b*) *Elastic Strain Energy.* When work is done by a force in straining an elastic spring of any kind, energy is transferred from some external source to the spring where it is stored as elastic strain energy which is a form of potential energy. The energy, thus stored, is measured by (*i.e.* numerically equal to) the work done in the transfer or in the straining of the spring, which may be the stretching or shortening or twisting of a helical coil of wire or it may be the bending or twisting of an elastic rod or beam.

If a force increasing, say, from zero to a maximum value P lb. at a displacement of l ft. is exerted in straining a spring, the work done and the energy stored in strained spring are equal to the average force multiplied by the displacement. And if the force is proportional to the displacement, as it usually is for an elastic spring, the average force is $\frac{1}{2}$P. Hence,

$$\text{Elastic strain energy} = \tfrac{1}{2}\text{P lb.} \times l \text{ ft.} = \tfrac{1}{2}Pl \text{ ft-lb.} \quad . \quad (2)$$

And if the force exerted, and equal opposing resistance, are at the rate of e lb. per ft. of displacement, $P = el$ and substituting this in (2),

$$\text{Elastic strain energy} = \tfrac{1}{2}el \times l = \tfrac{1}{2}el^2 \text{ ft.-lb.} \quad . \quad (3)$$

Similarly, if a torque rising from zero in proportion to the angular displacement to a maximum value T lb.-ft. at an angular displacement θ radians, strains a spring, the work done and the strain energy stored are equal to the angular displacement multiplied by the average torque,

which is $\frac{1}{2}$T when the angle of twist is proportional to the torque applied. Hence,

Elastic strain energy

$$=\tfrac{1}{2}\text{T lb.-ft.} \times \theta \text{ radians} = \tfrac{1}{2}\text{T}\theta \text{ ft.-lb.} \quad . \quad . \quad (4)$$

And corresponding to (3) this may be written,

Elastic strain energy

$$=\tfrac{1}{2}\theta \times (\text{torque per radian of twist} \times \theta)$$
$$=\tfrac{1}{2}\,(\text{torque per radian of twist}) \times \theta^2 \quad . \quad . \quad (5)$$

(*c*) *Kinetic Energy.* When a force is exerted, against the inertia resistance, in increasing the speed of a body, the energy transmitted is given to the body as energy of motion or kinetic energy. Similarly, when the speed of the body is reduced by an external resistance, kinetic energy is taken from the body.

If a particle of mass m moving with a velocity v_1 (or a body all parts of which move with the same linear velocity v_1), has exerted upon it a constant force P, in the direction of the velocity v_1, for a time t, then from equation (1) of Art. 7

$$\text{P}t = m(v_2 - v_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where v_2 is the velocity at the end of the time t. And since by Art. 5, the constant force will produce a constant acceleration, the average velocity will be $\frac{1}{2}(v_2+v_1)$ and the distance covered in the time t will be $s=\frac{1}{2}(v_2+v_1)t$. Hence the work done will be

$$\text{P}s = \frac{m(v_2-v_1)}{t} \times \tfrac{1}{2}(v_2+v_1)t = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \quad . \quad (7)$$

The quantity $\frac{1}{2}mv_2^2$ is the energy required to give the mass m the speed v_2 from rest (when $v_1=0$), and the quantity $\frac{1}{2}mv_2^2-\frac{1}{2}mv_1^2$ is the increase in kinetic energy when the speed of the mass m is increased from v_1 to v_2.

If the angular speed of a body about an axis be increased from ω_1 to ω_2 radians per second by the exertion of a torque T about the axis, then the work done will be

$$\text{T}\theta = \tfrac{1}{2}\text{I}\omega_2^2 - \tfrac{1}{2}\text{I}\omega_1^2 = \tfrac{1}{2}\frac{\text{W}}{g}k^2(\omega_2^2-\omega_1^2) \quad . \quad . \quad (8)$$

where θ is the angular displacement and I is the moment of inertia of the body (Art. 9) about the axis. The quantity $\frac{1}{2}I\omega_2^2$ or $\frac{1}{2}Wk^2\omega_2^2/g$ is the kinetic energy of angular motion, or of spin, about the axis at the speed ω_2, and $\frac{1}{2}\frac{Wk^2}{g}(\omega_2^2-\omega_1^2)$ is the increase in kinetic energy with increase of speed from ω_1 to ω_2 radians per second.

14. Power. Power is the rate of doing work or the rate of transmission of energy. The most obvious measure of power would be foot-pounds per second, but for historical reasons the common unit is the horse-power.

1 h.p. = 550 ft.-lb. per sec. = 33,000 ft.-lb. per minute . (1)

The relation of this to the electrical power units of the watt and the kilowatt is

1 h.p. = 746 watts = 0·746 kilowatt . . (2)

The watt is a rate of one joule per second, the joule, or 0·7374 ft.-lb., being the work done by a current of one ampere in one second at a potential difference of one volt.

15. Calculus Notation. Many of the foregoing definitions and relationships can be clearly and concisely stated in terms of the differential and integral calculus, which, where it is not a necessity, is a convenience and a simplification. Using the notation of the preceding articles, the following gives a summary in terms of the calculus, m being used in place of W/g.

Velocity $v = ds/dt$ or gradient of graph of s on base t (1)

at instant under consideration

Distance moved $s = \int_{t_1}^{t_2} v \,.\, dt$, or area under graph of v on base t . . . . . (2)

where t_1 and t_2 are the initial and final values of the **Time** t (t_1 can generally be made zero).

Angular velocity $\omega = d\theta/dt$, or gradient of $\theta - t$ graph . (3)

Angular movement

$$= \int_{t_1}^{t_2} \omega \, . \, dt, \text{ or area under } \omega - t \text{ graph} \quad . \quad . \quad (4)$$

Acceleration $$f = dv/dt = \frac{d\left(\frac{ds}{dt}\right)}{dt} = \frac{d^2s}{dt^2} \quad . \quad . \quad . \quad . \quad (5)$$

Final velocity $$v_2 = v_1 + \int_{t_1}^{t_2} f dt \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The last term being represented by the area under the graph of f on base t.

Angular acceleration $$\alpha = d\omega/dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Final angular velocity $$\omega_2 = \omega_1 + \int_{t_1}^{t_2} \alpha dt \quad . \quad . \quad . \quad . \quad . \quad (8)$$

For constant acceleration f, equation (6) becomes

Final velocity $$v_2 = v_1 + f(t_2 - t_1) \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and from equation (2), since

$$v = v_1 + ft \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$s = \int_{t_1}^{t_2} (v_1 + ft) dt = v_1(t_2 - t_1) + \tfrac{1}{2} f(t_2^2 - t_1^2) \quad . \quad (11)$$

or if t_1 be take as zero, and $t_2 = t$

$$s = v_1 t + \tfrac{1}{2} f t^2 \, . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and for constant angular acceleration α, equation (8) becomes

$$\omega_2 = \omega_1 + \alpha(t_2 - t_1) \, . \quad . \quad . \quad . \quad . \quad (13)$$

and the total angular motion θ in time t is

$$\theta = \omega_1 t + \tfrac{1}{2} \alpha t^2 \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Force $$\mathrm{P} = m \, . \, dv/dt = m \, . \, d^2s/dt^2 \quad . \quad . \quad . \quad (15)$$

$$\textbf{Impulse} = \int_{t_1}^{t_2} \mathrm{P} . dt = m \int_{t_1}^{t_2} f dt \text{ or } m \int_{t_1}^{t_2} \frac{dv}{dt} dt \text{ or } m \int_{t_1}^{t_2} \frac{d^2x}{dt^2} dt \quad (16)$$

Moment of Inertia of Mass $I = \int r^2 dm$ . . . . (17)

Moment of Inertia of Area $I = \int r^2 da$ . . . . (18)

Radius of Gyration of Mass $k^2 = \int r^2 dm / \int dm$ . . (19)

Radius of Gyration of Area $k^2 = \int r^2 da / \int da$ . . (20)

Angular Momentum $I\omega = Mk^2\omega = \int r^2 \omega dm$. (21)

Torque $T = I\alpha = \int r^2 \alpha dm = \alpha \int r^2 dm$ . . (22)

Work Total work of variable force $P = \int P \cos \theta \, . \, ds$, or area under a graph of $P \cos \theta$ on a base s . (23)

Work of variable torque $= \int T d\theta$ . . . . . (24)

Kinetic Energy of Rotation

$$\tfrac{1}{2} \int (r\omega)^2 dm = \tfrac{1}{2}\omega^2 \int r^2 dm = \tfrac{1}{2} I\omega^2 \, . \quad . \quad . \quad (25)$$

16. Vectors. A physical quantity which can be adequately specified by a number representing its magnitude on some scale of measurement is called a *scalar* quantity. Examples of scalar quantities are time, length, mass, temperature, and energy.

A quantity the complete specification of which involves a definite direction as well as magnitude is known as a *vector* quantity and can be completely represented by a vector. Examples of vector quantities are displacements, velocities, accelerations, and forces.

A *vector* is a straight line having definite length and direction, but not definite location in space.

17. Addition of Vectors. To find the sum of two vectors *ab* and *cd*, Fig. 1, set out *ab* of the given length and direction

and from the end b (on the right-hand side) set out be equal in length to and parallel to cd; join ae. Then ae is the geometrical or vectorial sum of ab and cd. We may write

$$ab + be = ae$$

or since be is equal to cd

$$ab + cd = ae.$$

This will be apparent if we take the given vectors as representing successive displacements of a point. A first displacement from a to b, followed by a displacement from

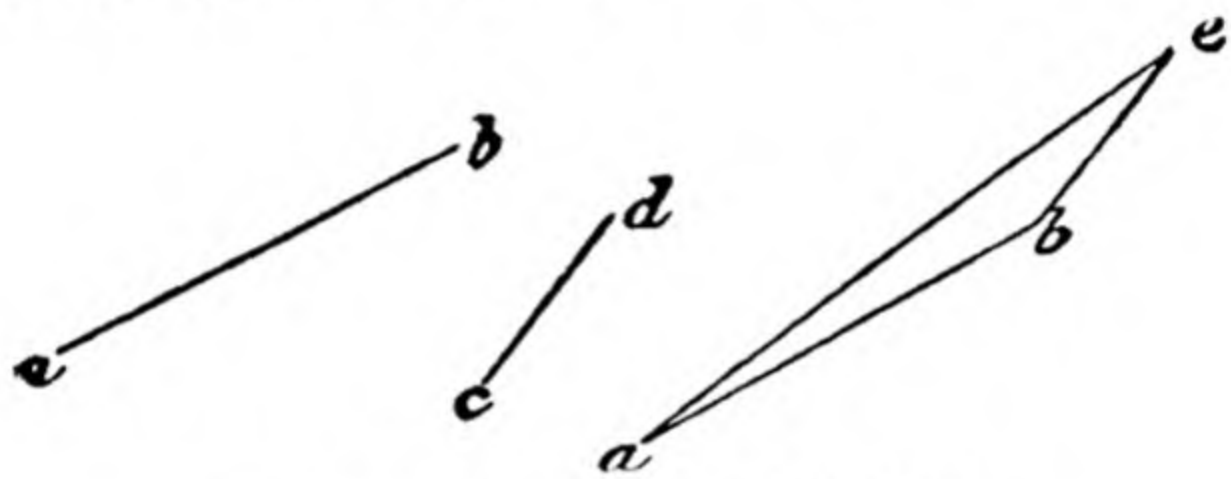

FIG. 1.—Addition of vectors.

b to e is exactly equivalent to a direct displacement from a to e. The same procedure would evidently apply if the displacements were those per second, *i.e.* velocities, and also holds for accelerations and for forces.

18. Subtraction of Vectors. If the vector cd, Fig. 2, is to be subtracted from the vector ab the result is found as for addition after the direction of cd has been reversed, that is dc is added to ab as shown in the triangle $a'b'c$ where $a'b'$ is drawn equal and parallel to ab, and from b', $b'e$ is drawn equal and parallel to dc. Then

$$a'e = a'b' + b'e = ab + dc = ab - cd.$$

The order of the added vectors is immaterial and the alternative form of triangle $d'c'f$ is shown at the right-hand side of the diagram (Fig. 2). Here $d'c'$ is equal and parallel to dc and from c' the vector $c'f$ is drawn equal and parallel to ab, so that

$$d'f = d'c' + c'f = dc + ab = -cd + ab.$$

If the two alternative vector triangles are placed so that $a'b'$ falls on $c'f$, they form a parallelogram of which the common side is a diagonal.

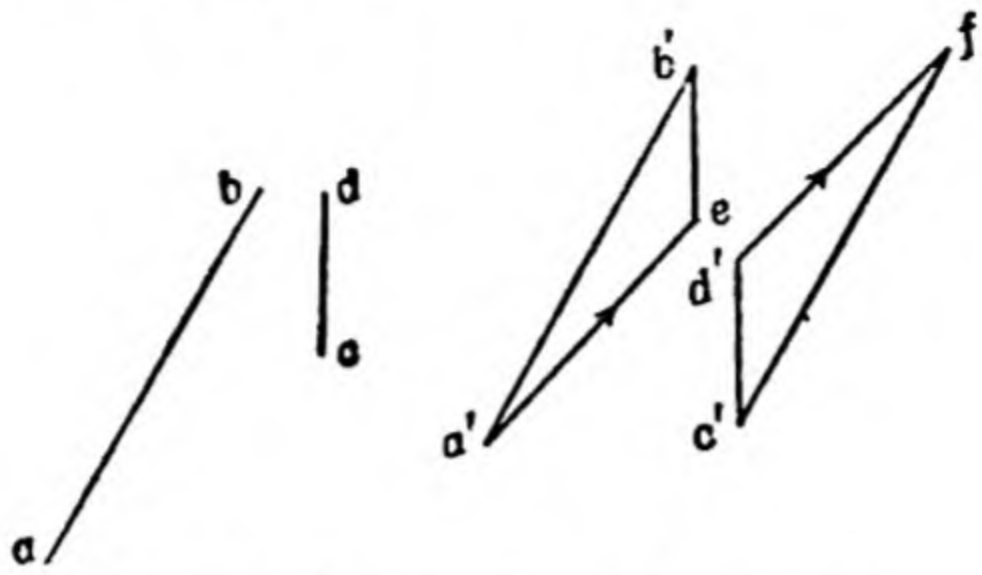

FIG. 2.—Subtraction of vectors.

19. Relative Motion. A point B is said to have motion relative to a point A if the line joining A to B alters in length or direction or in both. If the length alters, B has rectilinear motion, or motion of translation, relative to A. And if the direction AB alters, B has angular or rotatory motion about A. It may have both.

Although all motion is relative, it is convenient to regard

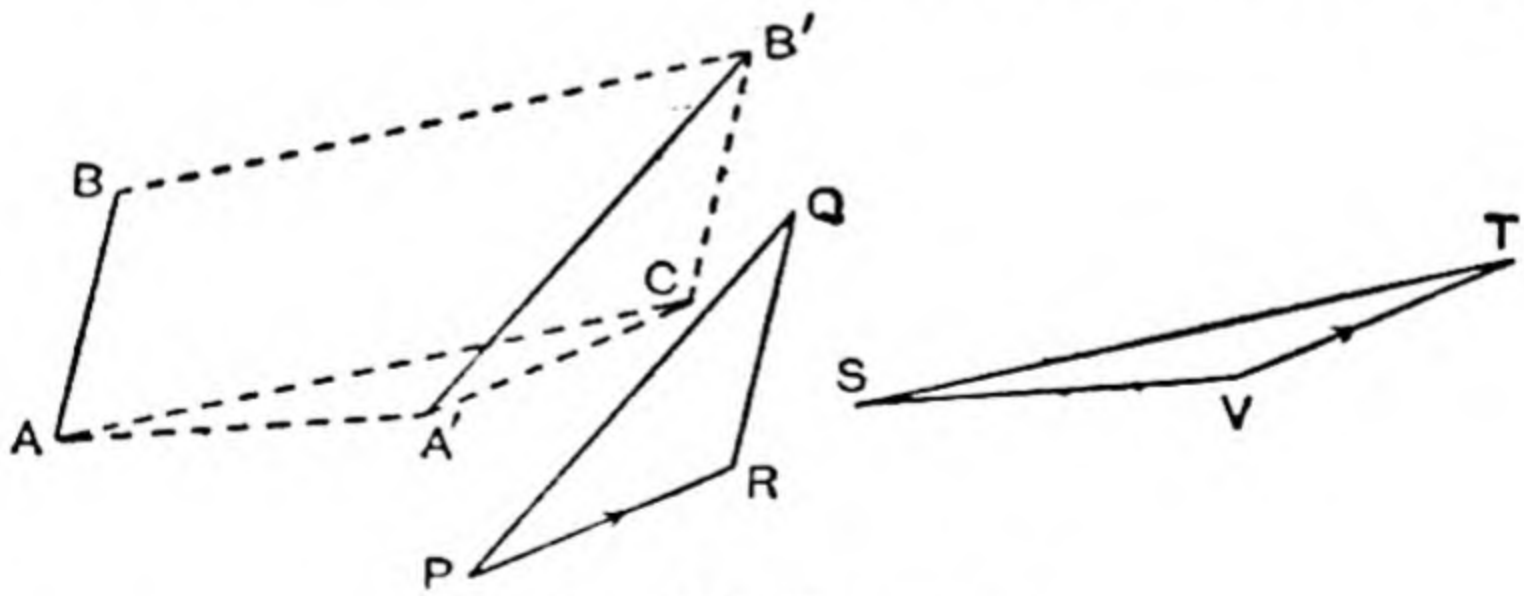

FIG. 3.—Relative motion.

some points as fixed, say relative to the earth, and to regard motion relative to such points as absolute and not relative motion.

If two points A and B, Fig. 3, have independent motion, in the plane of the diagram, to A′ and B′ respectively, the

displacement of B relative to A is vectorially BB′ minus AA′. If from any point P, a vector PQ is drawn to represent the final position of B relative to A, *i.e.* equal and parallel to A′B′, and QR is drawn parallel to AB and equal to it in length so that RQ represents vectorially the initial position of B relative to A, then PR represents the relative change of position, or the displacement, of B relative to A or

$$PR = PQ + QR = PQ - RQ = A'B' - AB \quad . \quad . \quad (1)$$

Alternatively the displacement of B relative to A may be found by drawing from any point S a vector ST equal and parallel to BB′ and a vector SV equal and parallel to AA′. Then

$$VT = VS + ST = -SV + ST = -AA' + BB' \quad . \quad (2)$$

The equivalence of the two constructions is apparent from the dotted lines BB′, AA′, A′C, and AC where BB′CA is a parallelogram and A′C is evidently equal (vectorially) to BB′ − AA′ and also to A′B′ − AB.

If the movements of A to A′ and B to B′ take place in the same interval of time, BB′ (or ST) represents the mean velocity of B, and AA′ (or SV) that of A, while VT, the vector difference, represents the mean velocity of B relative to A in the aforesaid interval of time.

It should be noticed that if A and B were points on a rigid body, their distance apart must remain constant or A′B′ = AB, and the only motion of B relative to A would be a rotation about A.

20. Polygon of Forces. If several forces are exerted on a body all through a common point, their resultant may be found by an extension of the principle of the addition of two vectors (Art. 17). Thus in Fig. 4, if forces Q, R, S, and T are exerted at P, then the resultant of Q and R may be found by drawing *ab* and *bc* parallel and proportional to Q and R respectively. Then a vector *ac*, which is the vector sum $ab + bc$ represents their resultant. Similarly, if *cd* is drawn parallel to S and representing by its length the magnitude of S to the scale employed for Q and R, then *ad*, the vector sum $ac + cd$ (and therefore also $ab + bc + cd$), represents the resultant of Q, R, and S. Similarly, drawing the vector *de* to represent T, the vector *ae* represents the

resultant of Q, R, S, and T, and this single force may, if convenient, replace the four forces.

If a fifth force represented by *ea* were exerted at the point P, the resultant of the five forces would be zero.

Thus an open-sided or unclosed force polygon gives the resultant force and a closed polygon of forces indicates a set of forces in equilibrium if the forces are concurrent, since they have a zero resultant. (They are not necessarily in equilibrium if not concurrent for they may then reduce to two equal and opposite forces not in line, *i.e.* to a couple, or torque, or turning moment.)

Conversely, if several forces are known to be in equilibrium, a polygon drawn with its sides parallel and proportional to the forces will form a *closed* figure.

If the closed figure is a triangle representing by its

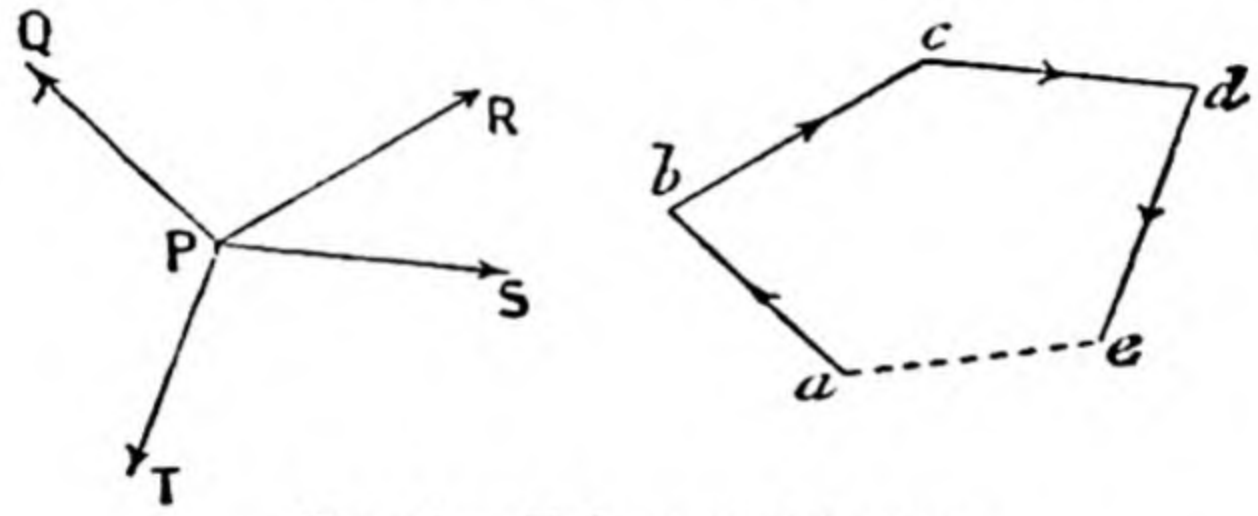

FIG. 4.—Polygon of forces,

sides three forces in equilibrium, it can have only one shape; but if the polygon has more than three sides, its shape will generally differ according to the order in which the vector sides representing the several forces are chosen.

Statically Indeterminate Problems. It may be noted that the open-sided polygon *abcde* may be closed by the single side *ea* of definite length and direction. It could also be closed by two intersecting sides, drawn in given directions which would be sufficient to determine the sides and the forces which they represent. But to close it by more than two sides, even in specified directions, would not determine the length of such sides, *e.g.* the polygon could be closed by three sides, in given directions, in an indefinitely large number of ways. Thus if a load were sustained by two supports (say ropes) in given directions, a vector diagram

would determine the supporting forces. But if it were supported by three ropes in the same plane and meeting at one point, a force polygon would not determine the pull in each rope. It would depend upon the size, material, and initial adjustment of the ropes, and such a problem is said to be statically indeterminate in contrast to one which can be solved by the principles of statics.

21. Resolution of Forces ; Rectangular Components. As an alternative to the use of vector diagrams drawn to scale, it is often convenient to resolve forces into two components at right angles to each other. The component forces in, say, a horizontal direction can then be added or subtracted algebraically instead of geometrically, and likewise the vertical components. If P, Fig. 5, be any force, by setting

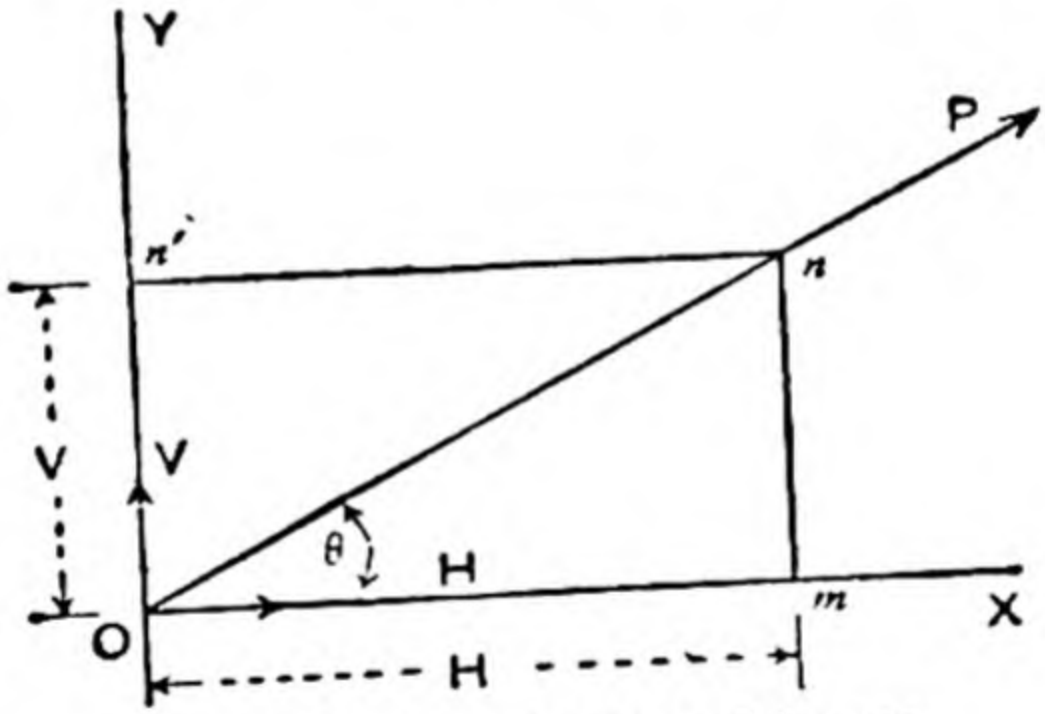

FIG. 5.—Resolution of forces.

off a length On to represent the magnitude of P and projecting On on the axes OX and OY we obtain the magnitudes of the horizontal component H represented by Om and the vertical component V represented by On'. And if θ is the inclination of On (the line of action of P) to OX, evidently

$$H = P \cos \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$V = P \sin \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Application of the process to several forces P_1, P_2, P_3, . . . etc., at angles θ_1, θ_2, θ_3, . . . etc., to OX, gives horizontal components H_1, H_2, H_3, etc., equal respectively

to $P_1 \cos \theta_1$, $P_2 \cos \theta_2$, $P_3 \cos \theta_3$, etc.; and if we add these algebraically, the total horizontal component H, of the several forces, and of their resultant is,

$$H = (H_1 + H_2 + H_3 + \text{etc.}) = (P_1 \cos \theta_1 + P_2 \cos \theta_2 + P_3 \cos \theta_3 + \text{etc.}) . \quad (3)$$

And similarly the total vertical component is

$$V = (V_1 + V_2 + V_3 \ . \ . \ . \ \text{etc.}) = (P_1 \sin \theta_1 + P_2 \sin \theta_2 + P_3 \sin \theta_3 \ . \ . \ . \ ) . \quad (4)$$

And evidently the resultant force R of P_1, P_2, P_3, etc., is

$$R = \sqrt{H^2 + V^2} = \sqrt{\{\Sigma(P \cos \theta)\}^2 + \{\Sigma(P \sin \theta)\}^2} . \quad (5)$$

and it acts at an angle ϕ to OX such that

$$\tan \phi = \frac{V}{H} = \frac{\Sigma(P \sin \theta)}{\Sigma(P \cos \theta)} \quad . \ . \ . \ . \quad (6)$$

It will be noticed that if the forces P_1, P_2, P_3, etc., are in equilibrium, R = O, which from (5) is only possible if

$$H = 0 \text{ and } V = 0 \quad . \ . \ . \ . \ . \quad (7)$$

or

$$\Sigma(P \cos \theta) = 0 \text{ and } \Sigma(P \sin \theta) = 0 \quad . \ . \quad (8)$$

these two statements (8) being equivalent to saying that the polygon is closed. And conversely, if H = 0 and V = 0, R is zero also, and the forces are in equilibrium if concurrent. (But not necessarily otherwise, for either H or V might be reduced to two equal and opposite forces not in line, having a zero vector sum but forming a couple and exerting a turning moment.)

EXAMPLES I

1. A chain 400 feet long and weighing 10 lb. per foot, hanging vertically from a drum, is wound up. Draw a diagram of the force required to draw it up when amounts, varying from 0 to 400 feet, have been wound up. From this diagram calculate the work done in winding up (*a*) the first 100 feet of the chain, (*b*) the whole chain.

2. A pit cage weighing 1000 lb. is suspended by a cable 800 feet long weighing 1½ lb. per foot length. How much work will be done in winding the cage up to the surface by means of the cable, which is wound on a drum?

3. In winding up a large clock (spring) which has completely "run down," 8½ complete turns of the key are required, and the torque applied at the finish is 200 lb.-inches. Assuming the winding effort is always proportonal to the amount of winding that has taken place, how much work has to be done in winding the clock? How much is done in the last two turns?

4. A shaft transmitting 50 h.p. runs at 80 revolutions per minute. Find the average twisting moment in pound-inches exerted on the shaft.

5. A flywheel weighing 2000 lb. has exerted upon it a torque of 500 lb.-ft. in excess of any frictional resistance. If its radius of gyration is 3 feet find its speed in revolutions per minute after it has made 8 complete revolutions from rest.

6. A wheel initially rotating at 120 revolutions per minute then has an acceleration of 2 radians per second per second. Through what angle will it turn in 0·5 second and in what time will it make one revolution?

CHAPTER II

STATICS

22. Equilibrium. The branch of mechanics relating to balanced systems of forces exerted on rigid bodies is called *statics*. If all the forces exerted on a body have a zero resultant and do not reduce to a couple (or torque) the forces are said to be balanced or in equilibrium, and sometimes the body is said to be in equilibrium. In many cases the body is stationary (or at rest) but not necessarily so. Thus a body in motion and subject to a number of forces which are in equilibrium, or balance, among themselves will continue in motion with unchanging velocity both linear and angular. Hence statics is not limited to bodies at rest but to bodies in unchanging motion, including the case of zero velocity or rest. This particular case is so common (*e.g.* it includes all structures) that statics might easily be thought to refer only to the mechanics of stationary bodies. The word is used as distinct from the kinetics or dynamics, the name of that portion of the subject which relates to bodies in accelerated motion, *i.e.* to bodies the velocities of which are changing in magnitude or direction or in both. (Chap. V.)

23. The Principle of Moments. If several forces in the same plane are exerted on a body, the moment of any two about any point in the plane is equal to the moment of their resultant. And adding a third force to the resultant of two, it follows that the moment of the resultant of three forces about the chosen point is equal to the moments of the three separate forces, and so on for any number of forces in the plane. If the forces are in equilibrium, their resultant is of zero magnitude; hence the total moment, or algebraic sum of the moments, of all the forces about any point in their plane is zero. In other words the total clockwise moments about any point are equal to the counter-clockwise moments about the same point. This fact gives a method of finding one or two unknown forces

of a number which maintain a body in equilibrium, particularly when a line of action or a point of application of an unknown force is known. When more than one force is unknown, an equation of clockwise and counter-clockwise moments about a point (such as a hinge centre) in the line of action of one of the unknown forces is most conveniently used to find the other unknown forces, since the moment of the first unknown force is zero about a point in its line of action and does not appear in the equation. A familiar example is found in the lever; moments about the fulcrum do not include any contribution from the supporting force at the fulcrum and give a single equation for the unknown effort to balance a known load.

The principle leads to the simple solutions of problems of finding the single equilibrant, or if sufficient information is given, of finding more than one equilibrating force of a system of parallel forces, *e.g.* the supporting forces exerted on a horizontal beam with vertical loads. Occasionally these supporting forces are regarded as components of a single parallel equilibrant, so that the principle of moments provides a method of resolution of parallel forces into components or of composition of components into a single resultant.

24. Couples. Two equal and opposite parallel forces not in the same straight line are not replaceable by a single resultant force and cannot be balanced by a single force. They constitute a *couple* and the perpendicular distance between them is called the arm of the couple. The moment of the couple is equal to the product of one of the two forces and the arm; it may be clockwise or counter-clockwise and is the same about any point in the plane of the two forces. A couple can be balanced by another couple of equal and opposite moment in the same plane, for the total moment of equal and opposite couples is evidently zero.

25. Reduction of Forces in One Plane. Several forces in one plane are statically equivalent to (1) a single resultant force, or (2) a couple, or (3) a system of forces in equilibrium (which can be regarded as a particular case of (1) where the resultant is of zero magnitude).

Any two forces, the lines of which intersect, may be

replaced by a single resultant equal to their vector sum acting at the point of intersection. And continuing the process of replacement of successive forces by compounding them with the resultants of others so far as possible, the system reduces to a single resultant, including the case of a zero resultant, or to a number of parallel forces. In the latter case, the parallel forces either reduce to a couple or are replaceable by a single parallel resultant equal in magnitude to the algebraic sum of the several parallel forces. Finally, the several forces must reduce to (1) a single resultant, or (2) a couple, or (3) zero resultant, the forces being then in equilibrium.

26. Conditions of Equilibrium of Several Forces All in One Plane. From the conclusion of Art. 25, if the forces reduce to a zero resultant, they are in equilibrium unless they form a couple (the moment of which is the same about every point in the plane of the forces). Therefore, in addition to there being a zero resultant, the only remaining necessary condition for equilibrium is that the forces shall have a zero moment about one point in the plane. Actually, if they are in equilibrium, their resultant moment about *every* point in the plane will be zero; but zero moment about one point is sufficient condition since a couple has the same moment about all points in its plane.

The necessary and sufficient conditions of equilibrium are three. They may be stated in various ways. (1) and (2) The vector force polygon must be a closed figure. This involves two conditions, such as the length and direction of the closing side (or equilibrant of the remainder). Alternatively the sum of the components in each of two independent directions must be zero, *e.g.* as in Art. 21.

$$\Sigma(\mathrm{P}\cos\theta)=0,\ \Sigma(\mathrm{P}\sin\theta)=0 \quad . \quad . \quad . \quad (1)$$

(3) The algebraic sum of the moments of all the forces about one point in the plane must be zero. In analytical form this might be stated as

$$\Sigma(x\mathrm{P}\sin\theta - y\mathrm{P}\cos\theta)=0 \quad . \quad . \quad . \quad . \quad (2)$$

where x and y stand for the co-ordinates of any point A on the line of action of a typical force P and $x\mathrm{P}\sin\theta - y\mathrm{P}\cos\theta$

gives its counter-clockwise moment about the origin O (Fig. 6). But we shall not need to use the condition in this mathematical form.

The three conditions provide for three relations among coplanar forces which are in equilibrium from which three unknown quantities may be found if all the remaining data are known, *e.g.* the unknown elements may be the magnitude and direction and position of a single equilibrating force, or the magnitude of one and the magnitude and direction of another of two unknown forces. An example of this occurs in Example I, Art. 27.

The three conditions may be stated in other ways. If the moments of the forces about each of three points not lying in a straight line are zero, the forces are in equilibrium. For the forces cannot reduce to a couple (which has a

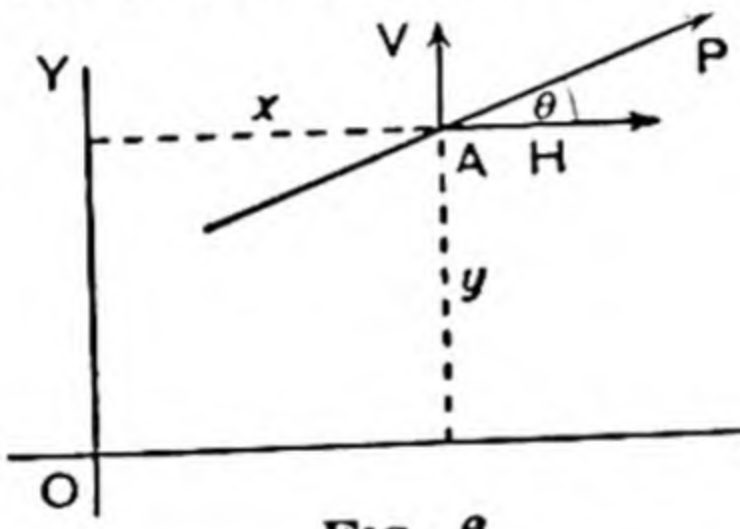

FIG. 6.

moment about *every* point in the plane) nor can it reduce to a single resultant, for though this could have a zero moment about two points if it passed through them, it cannot also have zero moment about a third point not in the line of the other two. Or again, if the forces satisfy the three conditions of having zero moments about two points in the plane and no total component in the direction of the line joining these two points, the forces are in equilibrium.

27. Solution of Statical Problems. When a rigid body has exerted upon it a number of forces all in one plane, some in the form of loads may be completely known and others which are supporting forces may be unknown or incompletely known; their positions may be known (*e.g.* by a hinged support) and the direction of one may be known (*e.g.* by a rope, chain, or other flexible suspension). In

such cases, the known conditions of equilibrium given in the preceding article (Art. 26) may often suffice to give the equations or graphical means by which the unknown elements in the supporting forces can be determined. The conditions are *three* and they serve to determine *three* elements of magnitude, direction, or position in one, two, or three incompletely known forces of a system of forces all in one plane.

When a number of rigid bodies are in contact, one with another, to form a system of rigid bodies, such as the several members of a girder or roof principal, the supporting forces of the structure as a whole may be found by these principles, which may further be applied to find the forces exerted upon the separate members, for each is itself a rigid body in equilibrium. For the composite rigid structure and for each part into which it is divisible, either actually or in imagination, the conditions of equilibrium are applicable, and we can write three equations for each by stating in algebraic form the three conditions of equilibrium. If we resolve the forces exerted upon the body (or part of it) in two mutually perpendicular directions, we can equate the components in opposite directions or the algebraic sums to zero, and we can equate the clockwise to the counter-clockwise moments about any one point or we can equate the algebraic sum to zero. We can thus proceed by simple algebraic equations with symbols for the unknown quantities, but later (Art. 33) we shall indicate alternative (graphical) methods of finding the unknown elements.

About *every* point in the plane of a system of coplanar forces in equilibrium, the total moment of all the forces is zero; and in some problems it may be more convenient to consider moments about two points and only the components in one direction, or to consider moments about three points and not resolve the force into components. By one selection or another, three independent relations can be established and equations for unknown quantities can be written. If more than three equations are formed, say, by taking moments about more than three points, they will be found to be not independent and those in excess of three will merely express again the relations already stated for three points.

A particular choice of direction for resolution into components is often more convenient than others, *e.g.* by resolving perpendicular to some unknown force, no component of the unknown force enters into the equation so formed. Also an unknown force may be eliminated in an equation of moments by taking the moments about some point in its line of action, about which it will have zero moment.

The application of the principles in the conditions of equilibrium is best appreciated by illustrations in concrete problems. Those which follow are simple, but the application of the principles is important because it shows how quite different looking problems may be solved by the same principles. The solution of statical problems by equations of components and of moments perhaps illustrates the principles of equilibrium better than the graphical methods, although the latter are in many cases very useful, whether the results are obtained by drawing to scale or by trigonometrical calculation from a diagram which is a mere sketch, not drawn to scale.

Smooth Bodies. In some problems in statics, contact between one body and another is at a smooth surface. In such cases, the force exerted by each body on the other is normal or perpendicular to the smooth surface, because there is no frictional component tangential to the smooth surface; hinges and rollers are common examples of such contact. No surface is absolutely smooth and frictionless, but many are smooth enough to make negligible the deviation from the normal of force transmitted from one body to another. A hinge may locate an unknown force and a roller or a flat smooth surface may not only locate a force but also give its direction if it contains the point of contact between two bodies.

EXAMPLE 1. A horizontal rod, 3 feet long, has a hole at one end, A, through which a horizontal pin passes forming a hinge. The other end, B, rests on a smooth roller at the same level. Forces of 7, 9, and 5 lb. act upon the rod, their lines of action, which are in the same vertical plane, intersecting it at distances of 11, 16, and 27 inches respectively from A, and making acute angles of 30°, 75°, and 45° respectively with AB, the first two sloping downwards

towards A, and the third sloping downwards towards B, as shown in Fig. 7. Find the magnitude of the supporting forces on the rod at A and B.

Since the end B rests on a smooth roller the reaction or supporting force R_B at B is perpendicular to the rod and

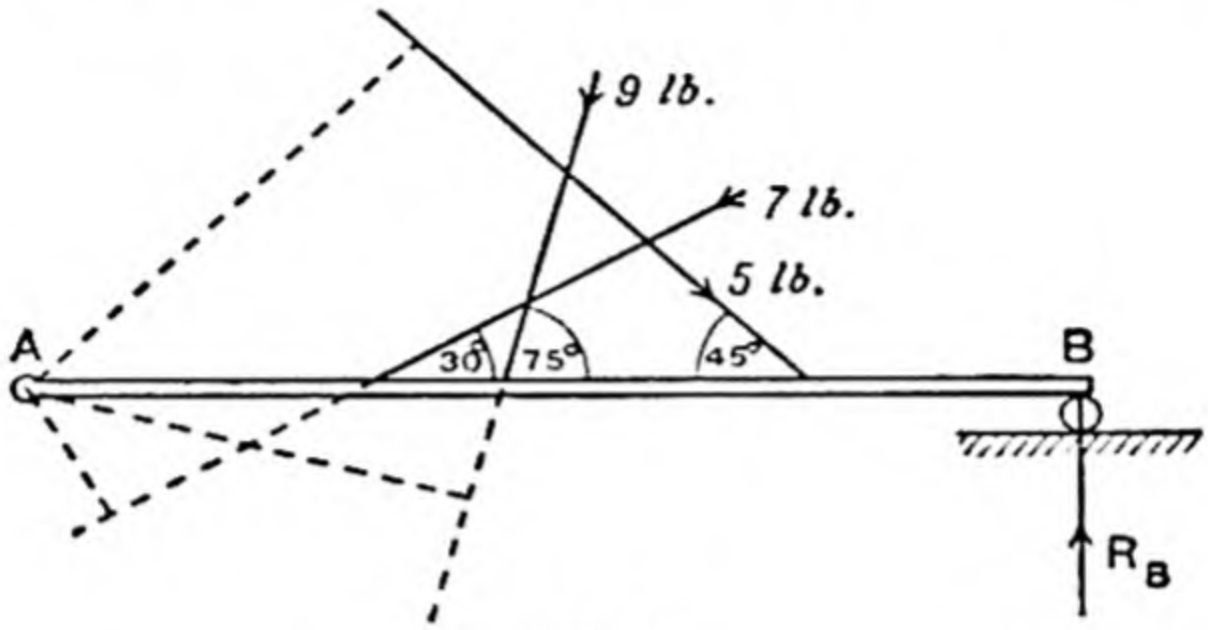

FIG. 7.

therefore vertical. We can most conveniently find this force by taking moments about A, because the only other force supporting the rod must act through this pin at A and has no moment about this point, so we shall have a simple equation with only one unknown force R_B.

The total clockwise moment about A is

$$7 \text{ lb.} \times (11 \sin 30°) \text{ in.} + 9 \text{ lb.} \times (16 \sin 75°) \text{ in.} + 5 \text{ lb.} \times (27 \sin 45°) \text{ in.}$$

$$= (77 \times 0{\cdot}5 + 144 \times 0{\cdot}966 + 135 \times 0{\cdot}707) \text{ lb.-in.}$$

$$= 273 \text{ lb.-in.}$$

The total counter-clockwise moment about A is $R_B \times 36$ in. Equating moments of opposite sign:

$$R_B \times 36 \text{ in.} = 273 \text{ lb.-in.}$$

$$R_B = \frac{273}{36} \frac{\text{lb.-in.}}{\text{in.}} = 7{\cdot}6 \text{ lb.}$$

We have used one condition of equilibrium to find R_B, viz. that the algebraic sum of moments about one point (A) is zero. There remain two more conditions to find the

two unknown items, namely, the magnitude and direction of R_A. Both could be found by drawing an open vector force polygon with sides representing the four now known forces 7, 9, 5, and 7·6 lb.; the closing side would represent R_A.

Or we could form the equations of opposite moments, about B and about some other point not in the line AB, but with an unknown direction of R_A as one of the two unknown quantities; this would lead to simultaneous equations with an angle as one unknown and would not be simple. Or, again, from moments about B, we could find directly the vertical component of R_A (since the horizontal component has no moment about B through which it passes), and then find the horizontal component by an equation of opposite moments about some other point not in the line AB.

The simplest way is to resolve the unknown R_A into horizontal and vertical components H_A and V_A, reckoned positive to the right and upwards, and apply the two conditions that the horizontal and vertical components of all the forces in equilibrium are each zero. Thus horizontally to the right in lb. units,

$$H_A - 7\cos 30° - 9\cos 75° + 5\cos 45° = 0$$
$$H_A = 7\times 0{\cdot}866 + 9\times 0{\cdot}259 - 5\times 0{\cdot}707 = 4{\cdot}86 \text{ lb.}$$

And upwards in lb. units,

$$V_A - 7\sin 30° - 9\sin 75° - 5\sin 45° + 7{\cdot}6 = 0.$$
$$V_A = 7\times 0{\cdot}5 + 9\times 0{\cdot}966 + 5\times 0{\cdot}707 - 7{\cdot}6 = 8{\cdot}13 \text{ lb.}$$

Compounding the components H_A and V_A as in (5), Art. 21,

$$R_A = \sqrt{(4{\cdot}86)^2 + (8{\cdot}13)^2} = 9{\cdot}47 \text{ lb.}$$

And from (6), Art. 21, the force is inclined to AB at an angle

$$\tan^{-1}(V_A/H_A) = \tan^{-1} 8{\cdot}13/4{\cdot}86 = \tan^{-1} 1{\cdot}67 = 59{\cdot}1°.$$

EXAMPLE 2. ABCD is a square, each side being 20 inches, and E is the middle point of AB. Forces of 7, 8, 12, 5, 9, and 6 lb. act on a body in the lines and directions AB, EC, BC, BD, CA, and DE respectively. Find the magnitude,

direction, and position with respect to ABCD, of the single force required to keep the body in equilibrium.

Note that here the three unknown elements all relate to a single force.

Let P be the required force, H_1 and V_1 its rectangular components in the directions AD and AB respectively (Fig. 8).

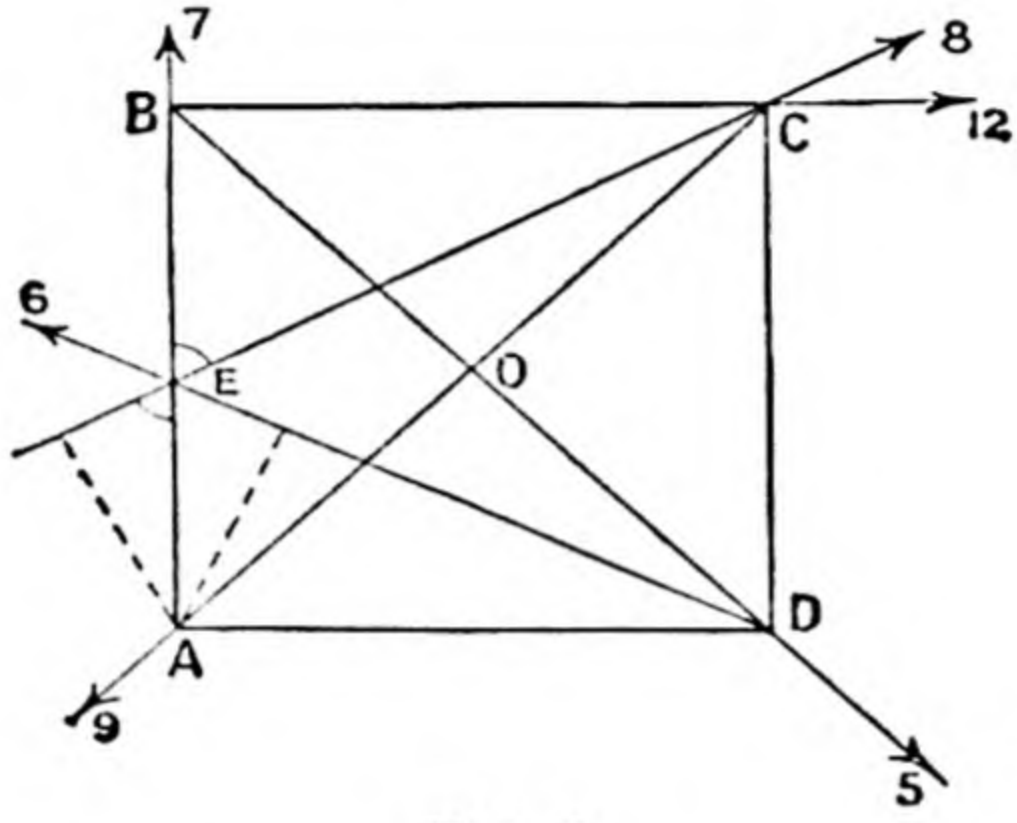

FIG. 8.

Resolving in direction AD and making H = 0 as in (7), Art. 21,

$$H = H_1 + 8 \cos B\widehat{C}E + 12 + 5 \cos 45° - 9 \cos 45° - 6 \cos E\widehat{D}A = 0$$

or $$H_1 + 8 \times 2/\sqrt{5} + 12 - 4 \times 0{\cdot}707 - 6 \times 2/\sqrt{5} = 0$$

$$H_1 = -12 - (2 \times 0{\cdot}894) + 4 \times 0{\cdot}707 = -10{\cdot}96 \text{ lb.}$$

Resolving in direction AB,

$$V = V_1 + 7 + 8 \cos C\widehat{E}B - 5 \cos 45° - 9 \cos 45° + 6 \cos A\widehat{E}D = 0$$

$$V_1 + 7 + 14 \times 1/\sqrt{5} - 14 \times 0{\cdot}707 = 0$$

$$V_1 = -7 - 6{\cdot}26 + 9{\cdot}90 = -3{\cdot}36 \text{ lb.}$$

Compounding H_1 and V_1,

$$P = \sqrt{(10{\cdot}96)^2 + (3{\cdot}36)^2} = 11{\cdot}46 \text{ lb.}$$

And is inclined to AD at an angle

$$\tan^{-1} \frac{-3{\cdot}36}{-10{\cdot}96} = \tan^{-1} 0{\cdot}3066 = 180° + 17° = 197°.$$

Not 17°, since both H_1 and V_1 are negative.

The magnitude and direction of the equilibrant P are now known but not its position; unless it is in the same straight line as the resultant of the original forces (to which it is equal and opposite), it will form with it a couple. The remaining necessary condition for equilibrium is that the equilibrant P shall have about one point in the plane of the forces a moment equal and opposite to that of the other forces (or of their resultant). By choosing the point A we shall eliminate the moments of the forces of 9 and 7 lb. which pass through A.

The total clockwise moment about A of the original forces in in.-lb. is

$$12 \times AB + 8 \times AE \sin B\hat{E}C + 5 \times OA - 6 \times AE \sin A\hat{E}D$$
$$= 12 \times 20 + 8 \times 10 \times 2/\sqrt{5} + 5 \times 20 \times 0{\cdot}707 - 6 \times 20 \times 1/\sqrt{5}$$
$$= 240 + 160/\sqrt{5} + 70{\cdot}7 - 120/\sqrt{5} = 328{\cdot}6 \text{ lb.-in.}$$

To balance this, the force (P) of 11·46 lb. must exert a counter-clockwise moment of 328·6 lb.-in. If a is its leverage arm about A,

$$11{\cdot}46 \text{ lb.} \times a = 328{\cdot}6 \text{ lb.-in.}$$
$$a = 328{\cdot}6 \text{ lb.-in.}/11{\cdot}46 \text{ lb.} = 28{\cdot}7 \text{ in.}$$

The position of the force is shown in Fig. 9. Its line of action touches an arc of 28·7 in. radius about A as centre; and since it has to exert a counter-clockwise moment about A, it passes above A. Its position might also be specified as cutting AB produced at E a distance 28·7/cos 17° = 30 inches from A.

28. Method of Sections. The principles of the preceding article may be applied to find the forces transmitted by the members of a structure consisting of, say, tie bars and

struts jointed together by hinges to form a frame such as a bridge girder or a roof principal. For not only is the

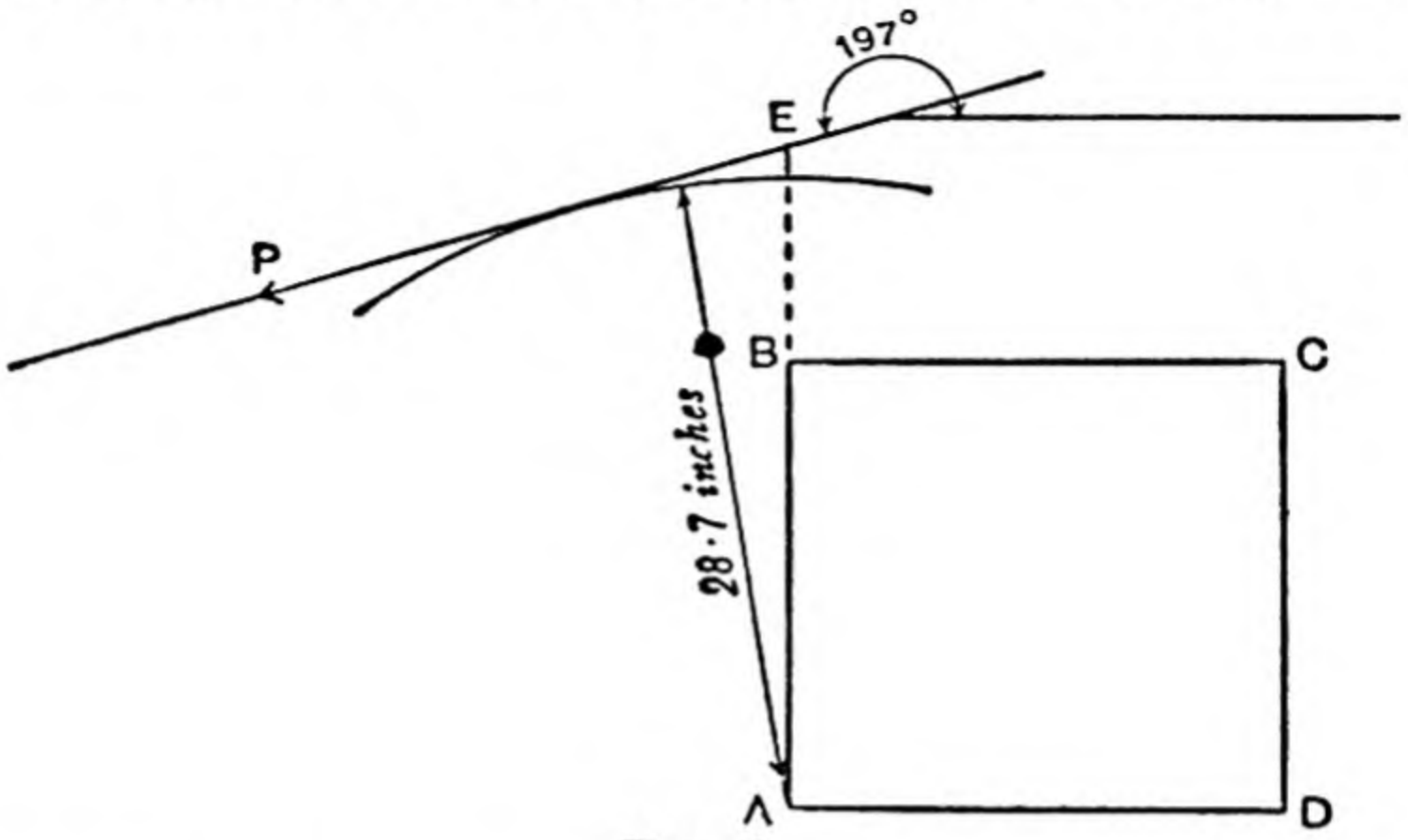

FIG. 9.

structure in equilibrium under the action of loads and supporting forces but every member is in equilibrium. If the structure is divided by an imaginary plane of section into two parts, each may be treated as a body in equilibrium

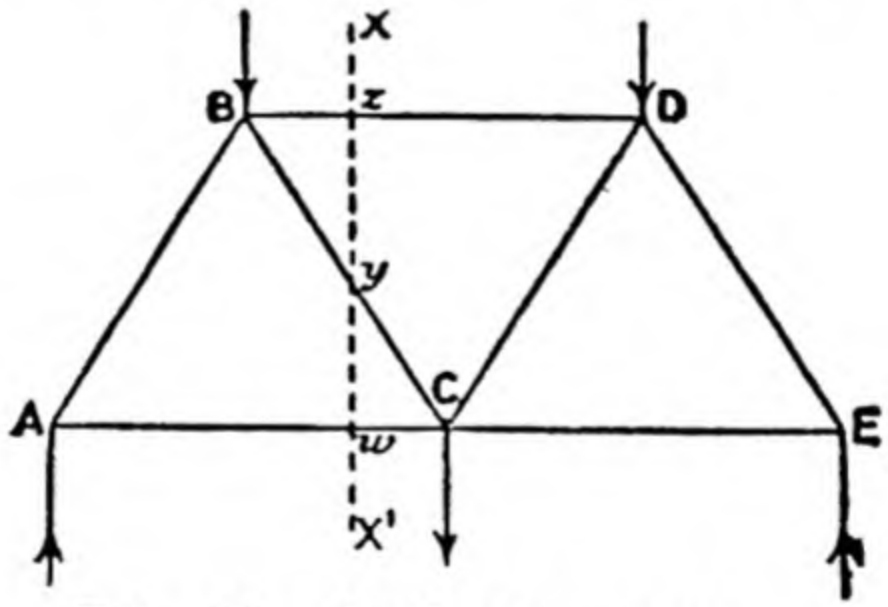

FIG. 10.—Method of sections.

under certain forces, some of which are exerted by members cut by the plane of section.

For example, if a hinged frame such as ABCDE, Fig. 10. is in equilibrium under the action of the loads, say, W_B,

W_C, and W_D at B, C, and D respectively, and the upward vertical supporting forces R_A, R_E at A and E respectively, and an imaginary plane of section XX′ perpendicular to the plane of the structure be taken, then the portion AB*zyw* is in equilibrium under the forces at A and B and the forces exerted on it by the remainder of the structure through the bars BD, BC, and AC. Of these (BD and AC being thin horizontal rods), only BC can transmit any vertical force. Take all the bars to be of equal length and therefore all the internal angles to be 60°. Then resolving vertically upward, and equating to zero, all the forces exerted on this portion AB*zyw* of the structure, if P is the axial pull exerted at *y* by the bar BC on the portion of the structure to the left of XX′.

$$R_A - W_B - P\cos 30° = 0$$

$$P = (R_A - W_B)/\cos 30°$$

$$= (R_A - W_B)2/\sqrt{3}$$

If W_B exceeds R_A, P will be negative, *i.e.* P will be a thrust, and BC will be a strut and not a tie-bar. Similarly, if the plane of section is taken between A and B, equating vertical forces to the left of the plane,

$$\text{Downward thrust in AB} \times \cos 30° = R_A$$

$$\text{Thrust in AB} = R_A/\cos 30° = 2R_A/\sqrt{3}$$

And by equating the two opposite horizontal forces on this portion of the structure (to the left of B), it is clear that the horizontal component of the thrust exerted upon it through AB must just balance the force exerted on it through the bar AC. If T is the tension in this bar,

$$T = \text{Thrust in AC} \times \cos 60° = \tfrac{1}{2} \times 2R_A/\sqrt{3} = R_A/\sqrt{3}$$

Or we might have taken the vertical section through B and considered the equilibrium of the structure to the left of it (or simply of the bar AB), and the condition that the moment of all forces on it must balance. The only forces having any moment about B are T and R_A, hence

$$T \times AB \sin 60° = R_A \times AB \cos 60°$$

$$T = R_A \cot 60° = R_A/\sqrt{3}.$$

If in any case we assume a force to be a pull when it is actually a thrust, the result would be a negative quantity, thus giving a correct result.

This method of sections is a simple one, particularly for finding the force transmitted by one or two members of a structure without having to find the forces in all members.

EXAMPLE. One end of a girder made up of bars jointed together is shown in Fig. 11. Vertical loads of 3 tons and 5 tons are carried at B and C respectively, and the vertical supporting force at H is 12 tons. The sloping bars are inclined at 60° to the horizontal. Find the forces in the bars CD, CE, and FE.

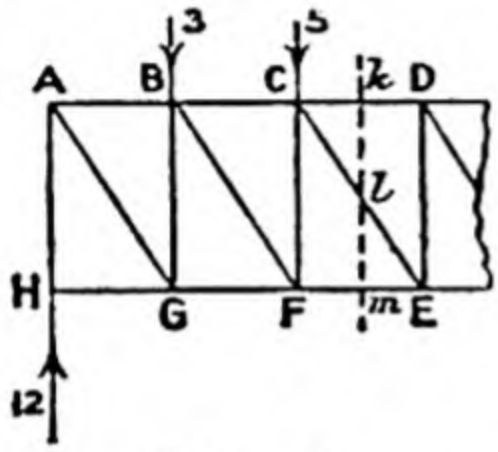

FIG. 11.

The portion of the girder ACFH cut off by the vertical plane *klm* is in equilibrium under the action of the loads at B and C, the supporting force at H, and the forces exerted by the bars CD, CE, and FE on the joints at C and F. Resolving these forces vertically, the forces in CD and FE have no vertical component, hence the *downward* vertical component force exerted by CE on the left-hand end of the girder is equal to the excess upward force of the remaining three, *i.e.* $12-3-5=4$ tons; hence

$$\text{Force in CE} \times \cos 30° = 4 \text{ tons}$$

$$\text{or force in CE} = 4 \times 2/\sqrt{3} = 4{\cdot}62 \text{ tons}$$

This, being positive, acts downwards on the portion ACFH, *i.e.* it acts from *l* towards E, and the bar CE *pulls* at the joint C, hence the bar CE is in tension to the amount of 4·62 tons. To find the force in bar FE, take a vertical section plane through C or indefinitely near to C, and just on the right hand of it. Consider the equilibrium of the part of the girder to the left of this section, and taking moments about C and reckoning clockwise moments positive,

$$12 \times \text{AC} - 3 \times \text{BC} + (\text{force in FE}) \times \text{FC} = 0$$

$$12 \times 2 - 3 \times 1 + (\text{force in FE}) \times \sqrt{3} = 0$$

$$\text{and force in FE} = -21/\sqrt{3} = -12{\cdot}12 \text{ tons}$$

The negative sign indicates that the force in FE acts on F in the opposite direction to that in which it would have a clockwise moment about C, *i.e.* the force *pulls* at the joint F; hence the member is in *tension* to the extent of 12·12 tons.

Similarly, taking a section just to the left of E, considering the equilibrium of the part of the girder to the left of this section, and taking, say, clockwise moments about E, the force in CD is found to be a push of 14·43 tons towards C, *i.e.* CD has a compressive force of 14·43 tons in it, as follows:

$$12\times3-3\times2-5\times1+\sqrt{3}\,(\text{force in CD})=0$$

$$\text{force in CD}=-14{\cdot}43 \text{ tons.}$$

29. Equilibrium of Three Forces. If a body is in equilibrium when three forces are exerted upon it, the three forces must either be all parallel or all pass through a common point, or in other words they must be concurrent. For unless all three forces are parallel, the lines of action of two must meet and these two forces are replaceable by a single resultant acting through their point of intersection. This resultant cannot be balanced by the third force unless equal and opposite to it and in the same straight line, in which case the third force passes through the intersection of the other two and the three forces are concurrent.

This concurrency of three forces simplifies some problems of statics by fixing the direction of an unknown force, since its line of action passes through those of the other two, which may be known, at their intersection. The magnitudes of the forces can be found from a vector force triangle or by resolution into rectangular components.

Other problems of plane statics involving more than three forces in balance may often be reduced to that of the equilibrium of three forces. This may be done by replacing two known forces by their resultant at their point of intersection and extending such replacements until the forces are reduced to three.

EXAMPLE 1. A ladder 18 feet long rests with its upper end against a smooth vertical wall, and its lower end on rough ground 7 feet from the foot of the wall. The weight

of the ladder is 60 lb., which may be looked upon as a vertical force halfway along the length of the ladder. Find the magnitude and direction of the forces exerted by the wall and the ground on the ladder.

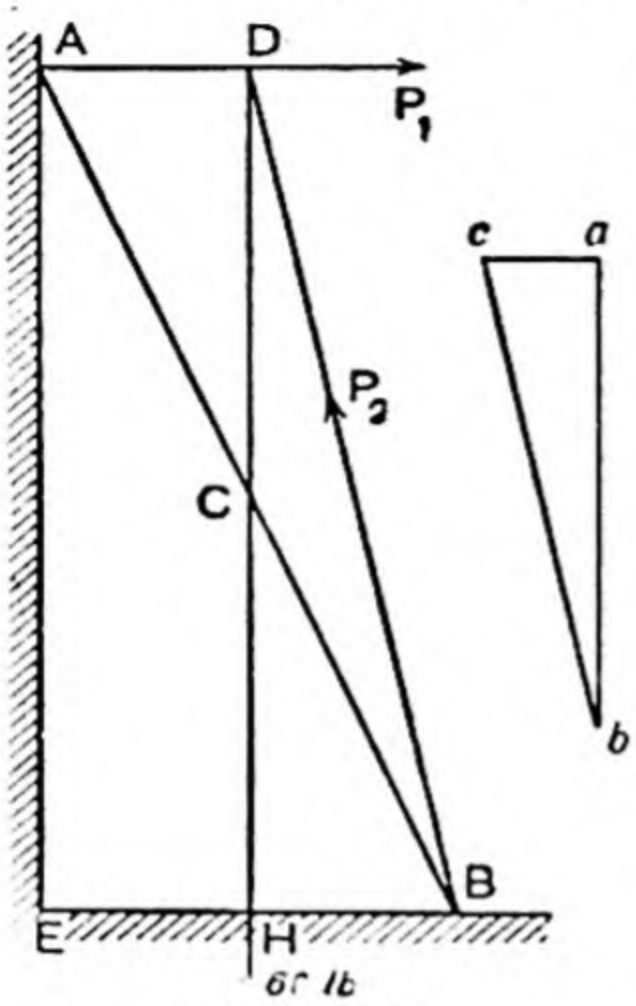

FIG. 12.

The weight of 60 lb. acts vertically through C (Fig. 12), and the reaction P_1 of the wall, is perpendicular to the wall since the latter is smooth. These two forces intersect at D. The only other force on the ladder is the thrust P_2 which the ground exerts on it at B. This third force must therefore act through D since the three forces are necessarily concurrent and its line of action must be along BD. The magnitude of P_1 may be found by equating opposite moments about B, thus avoiding any moment of P_2. Thus

$$P_1 \times AE = 60 \times \tfrac{1}{2}BE$$

$$P_1 \times \sqrt{18^2 - 7^2} = 60 \times 3{\cdot}5 \text{ lb.-ft.}$$

$$P_1 = 210/\sqrt{275} = 12{\cdot}67 \text{ lb.}$$

And since P_2 balances the horizontal force of 12·67 lb. and the vertical force of 60 lb., its components are known; hence

$$P_2 = \sqrt{(12{\cdot}67)^2 + (60)^2} = 61{\cdot}4 \text{ lb.}$$

and P_2 is inclined to EB at an angle

$$\tan^{-1} AE/\tfrac{1}{2}EB = \tan^{-1} (\sqrt{275}/3{\cdot}5) = \tan^{-1} 4{\cdot}75 = 78{\cdot}1°.$$

A second method of solving the problem consists in drawing a vector triangle, *abc* (Fig. 12), representing by its vector sides P_1, P_2, and 60 lb. The 60-lb. force *ab* are set off to scale, and *bc* and *ca* drawn parallel to P_2 and P_1 respectively, and the magnitudes are then measured to the

same scale. A third method consists (without drawing to scale) of solving the triangle *abc* trigonometrically, thus

$$P_1 : P_2 : 60 = ca : cb : ab$$
$$= HB : BD : HD$$
$$= 3{\cdot}5 : \sqrt{\{(3{\cdot}5)^2 + 275\}} : \sqrt{(275)}$$

From which,

$$P_1 = 60 \times 3{\cdot}5 / \sqrt{275} = 12{\cdot}67 \text{ lb.}$$

and

$$P_2 = 60 \times \sqrt{287} / \sqrt{275} = 61{\cdot}3 \text{ lb.}$$

EXAMPLE 2. A light bar, AB, 20 inches long, is hinged at A so as to be free to move in a vertical plane. The end B is supported by a cord, BC, so placed that the angle $A\widehat{B}C$ is 145° and AB is horizontal. A weight of 7 lb. is hung on the bar at a point D in AB, 13 inches from A. Find the tension in the cord and the pressure of the rod on the hinge.

Let T be the tension in the cord (*i.e.* the pull which the cord exerts on the bar) and P be pressure exerted by the

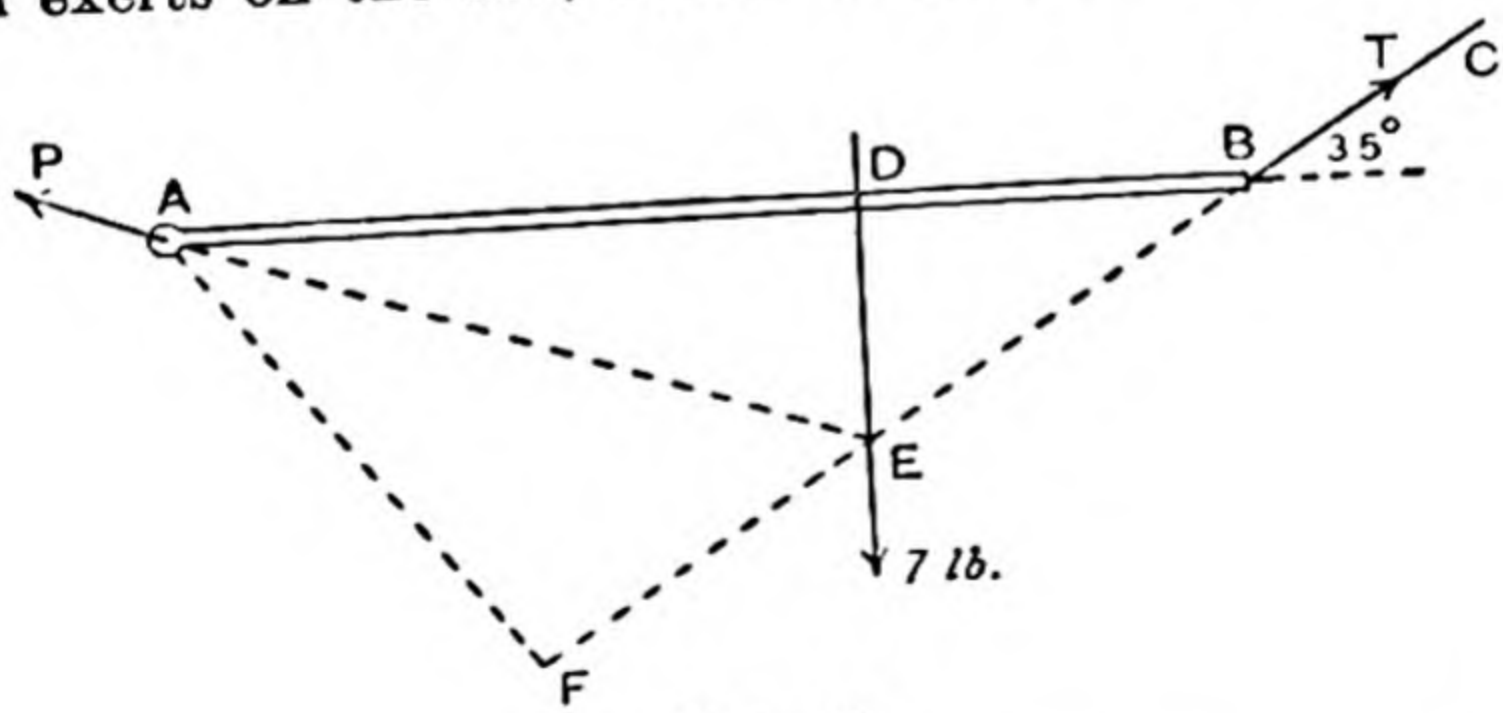

FIG. 13.

hinge at A on the bar. Then P must pass through the intersection E, Fig. 13, of the lines of action of T and the weight of 7 lb. at D; hence EA gives the direction and position of P. To find T take moments about A (since P has no moment about this point),

$$T \times AF = 7 \text{ lb.} \times AD$$
$$T \times 20 \sin 35° = 7 \text{ lb.} \times 13 \text{ in.}$$
$$T = 91/11{\cdot}47 = 7{\cdot}93 \text{ lb.}$$

The vertical upward component of P is $(7 - T \sin 35°) = 2{\cdot}45$ lb. The horizontal component is $T \cos 35° = 6{\cdot}5$ lb.

$$P = \sqrt{(6{\cdot}5)^2 + (2{\cdot}45)^2} = 6{\cdot}95 \text{ lb.}$$

Its inclination to the horizontal,

$$\widehat{DAE} = \tan^{-1}(2{\cdot}45/6{\cdot}5) = 20° \ 40'.$$

The pressure of the bar on the hinge is then 6·95 lb. in an opposite direction, *i.e.* AE, 20° 40′ downwards. Example 1, Art. 27, might be similarly solved by finding the resultant of the known forces. This and the two supporting forces must be concurrent.

30. Balance of Coplanar Forces by Three of Unknown Magnitude. The three conditions of equilibrium allow three unknown elements to be found relating to a number of forces in one plane which are in equilibrium. The three unknown elements may be the magnitude, direction, and position of a single equilibrant (as in Example 2 of Art. 27) or the magnitude of two and the direction of one of them (as in Example 1, Art. 27). A remaining case of some practical interest is that in which the three unknown elements are the magnitudes of three forces, the positions and directions of which are known. (It may be noted that it is not always possible to balance a number of forces by other forces in three given lines of action, *e.g.* horizontal forces cannot be balanced by vertical forces unless they form a couple.) Methods of solving such problems may best be given by an example.

Example 1. Suppose a light stiff horizontal bar, Fig. 14, to be supported at points w, y, and z by cords inclined at the angles shown in the diagram and carries at x a vertical load of 50 lb. What are the tensions T_1, T_2 and T_3 in the cords attached to w, y and z respectively, the distances wx, xy and yz each being 8 ft.

If the lines of action of T_1 and the 50 lb. load are produced to meet in B, and those of T_2 and T_3 to meet in A, it is

evident that T_1 and the 50 lb. force may be replaced by their resultant at B and that T_2 and T_3 may be replaced by by their resultant at A. The forces are thus reduced to two which are in equilibrium. Consequently they must be equal and act in opposite directions in the same straight

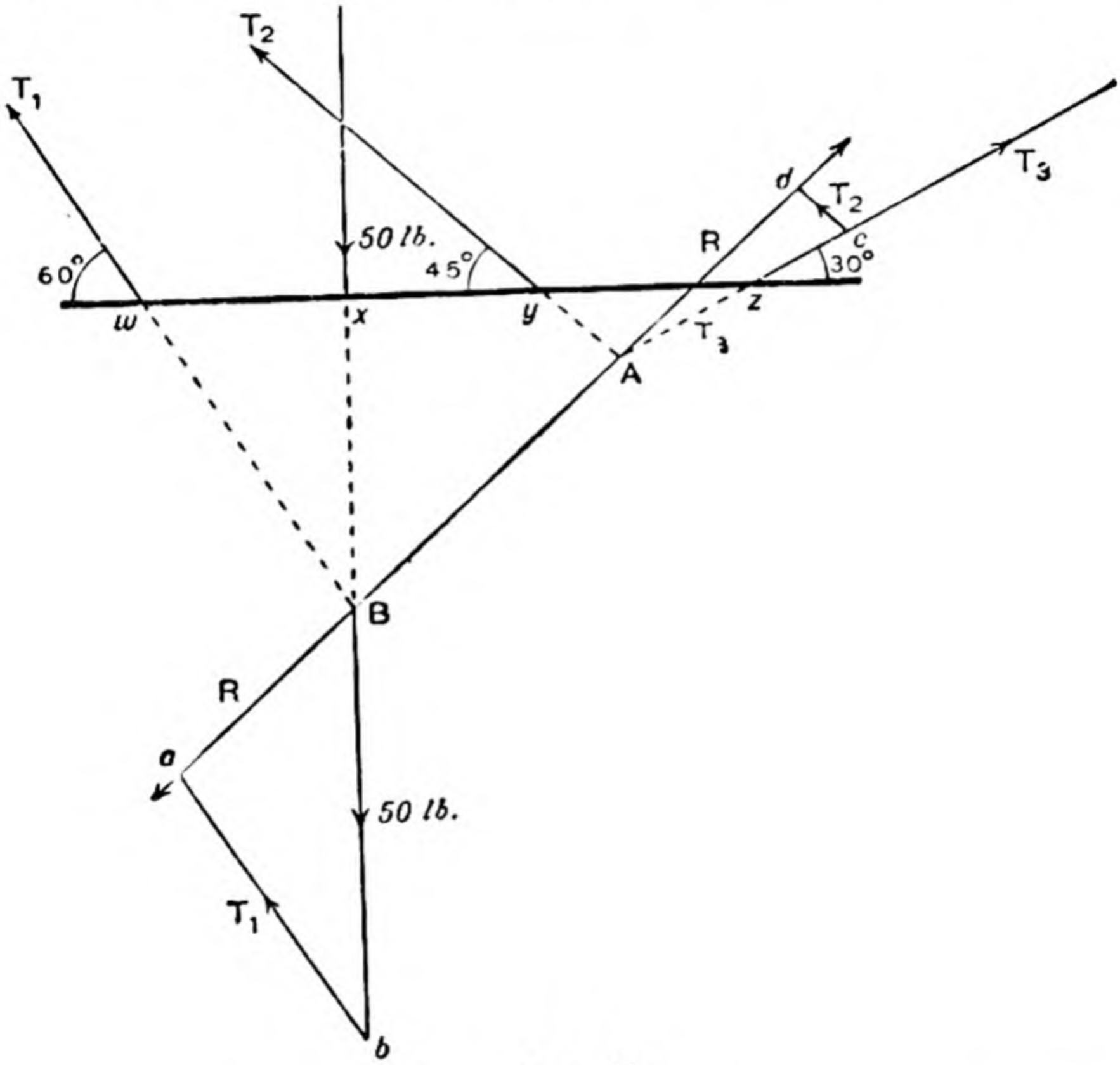

FIG. 14.

line; and since A and B are points in their common line of action, both must act along AB but in opposite directions. Let P denote their common magnitude. If the vector B*b* be drawn vertically to represent 50 lb. to scale and *ba* be drawn parallel to the line of T_1, to meet the line AB

produced at a, the vector ba represents T_1, for vectorially $Bb+ba=Ba$, and Ba represents the equilibrant R in the direction BA. And if the vector Ad equal in length to Ba be set off in BA produced, and dc be drawn parallel to T_2 to meet the line of T_3 in c, the magnitudes of T_2 and T_3 are represented to scale by the lengths of cd and Ac respectively. Scaling the magnitudes from the diagram, it is found that $T_1=36{\cdot}6$ lb., $T_2=6{\cdot}9$ lb. and $T_3=26{\cdot}9$ lb.

The results could also conveniently be found by numerical calculation of moments if the arms or leverage distances are scaled from the diagram or are calculated trigonometrically. Thus if we consider moments about A, since T_2 and T_3 have no moments about this point, evidently the moments of T_1 and the 50 lb. load about A must be equal and opposite. Since cA is perpendicular to the line of T_1, the distance of A from T_1 is equal to the distance of T_1 from z minus Az. And

$$Az=yz \sin 45°/\sin 105°=8\times 0{\cdot}707/0{\cdot}9659=5{\cdot}86 \text{ ft.}$$

The distance of z from $T_1=wz\cos 30°=24\times\sqrt{3}/2=20{\cdot}78$ ft. Distance of A from $T_1=20{\cdot}78-5{\cdot}86=14{\cdot}92$ ft. Distance of A from line of 50 lb. force $=8+Az \sin 30°=8+\frac{1}{2}$ of $5{\cdot}86=10{\cdot}93$ ft.

Equating moments of T_1 and 50 lb. force about A,

$$T_1\times 14{\cdot}92 \text{ ft.}=50 \text{ lb.}\times 10{\cdot}93 \text{ ft.}$$
$$T_1=36{\cdot}6 \text{ lb.}$$

Equating opposite moments about z,

$$36{\cdot}6 \text{ lb.}\times 20{\cdot}78 \text{ ft.}+T_2\times 8\times\sin 45°=(50\times 16) \text{ lb.-ft.}$$
$$5{\cdot}656\ T_2=800-760{\cdot}7$$
$$T_2=6{\cdot}95 \text{ lb.}$$

Equating moments about y in lb.-ft.,

$$36{\cdot}6\times 16\cos 30°-50\times 8=T_3\times 8\sin 30°$$
$$4T_3=36{\cdot}6\times 13{\cdot}86-400=107{\cdot}1$$
$$T_3=26{\cdot}8 \text{ lb.}$$

The numerical calculations may be checked by verifying that the total vertical and horizontal components of the forces are each zero; thus—

Vertical component

$$=36{\cdot}6\times\sqrt{3}/2+6{\cdot}95\times0{\cdot}707+26{\cdot}8\times0{\cdot}5-50=0{\cdot}07 \text{ lb.}$$

Horizontal component

$$=36{\cdot}6\times0{\cdot}5+6{\cdot}95\times0{\cdot}707-26{\cdot}8\times\sqrt{3}/2=0{\cdot}01 \text{ lb.}$$

which verifies the previous computations (within 0·1 lb.).

In the foregoing example only one force or load was given, but if there were two or more it would only be necessary to find their resultant and then to treat the resultant as the single known force. In many problems the moments may best be calculated from measurements of the arms of the moments to scale.

It may be observed that the solution is an application of the three-force theorem of Art. 29. For T_1 and the 50 lb. force must have resultant R through the point B. Then R, T_2, and T_3 are three forces in equilibrium; hence, since they are not parallel, they are concurrent, and R must pass through A the intersection of T_2 and T_3, *i.e.* it must act along the line AB.

The choice of intersections for use in the solution is not the only possible one. The intersection of either T_2 or T_3 with the 50 lb. force might have been joined to the intersection of T_1 with the remaining force to give the line of action of two equal and opposite forces.

EXAMPLE 2. A truck shown in Fig. 15 is to be hauled by the horizontal force P up the step or obstacle shown touching a front wheel. Find the force P when the truck begins to mount the step. Find also the reactions of the step on the front wheel and the ground on the back wheel. The combined weight of truck and load is W in the line shown. Neglect the friction at the axles. At what angle would a pull applied at C be least in order to move the truck?

The solution of this problem follows the lines of that of Example 1, and only needs brief indication by the broken lines in Fig. 15. Since the reactions R_1 and R_2 of the step and the ground must pass through the axles, their directions are as shown and their lines of action meet in A_1, while the lines of action of P and W meet in B_1. The resultant of

R_1 and R_2 and that of P and W balance in the line A_1B_1, and two vector triangles will determine P, R_1, and R_2 in magnitude when W is given. An alternative pair of points of intersection is shown at A_2B_2. But if R_1 and P are paired, meeting at A_3, the two remaining forces R_2 and W, being parallel, do not intersect; their resultant must be vertical and equal to $W-R_2$. Its position could be found by equating the moment of $W-R_2$ about any point, to the algebraic sum of the moments of W and R_2 about the same point. But the resultant of R_1 and P to balance that of

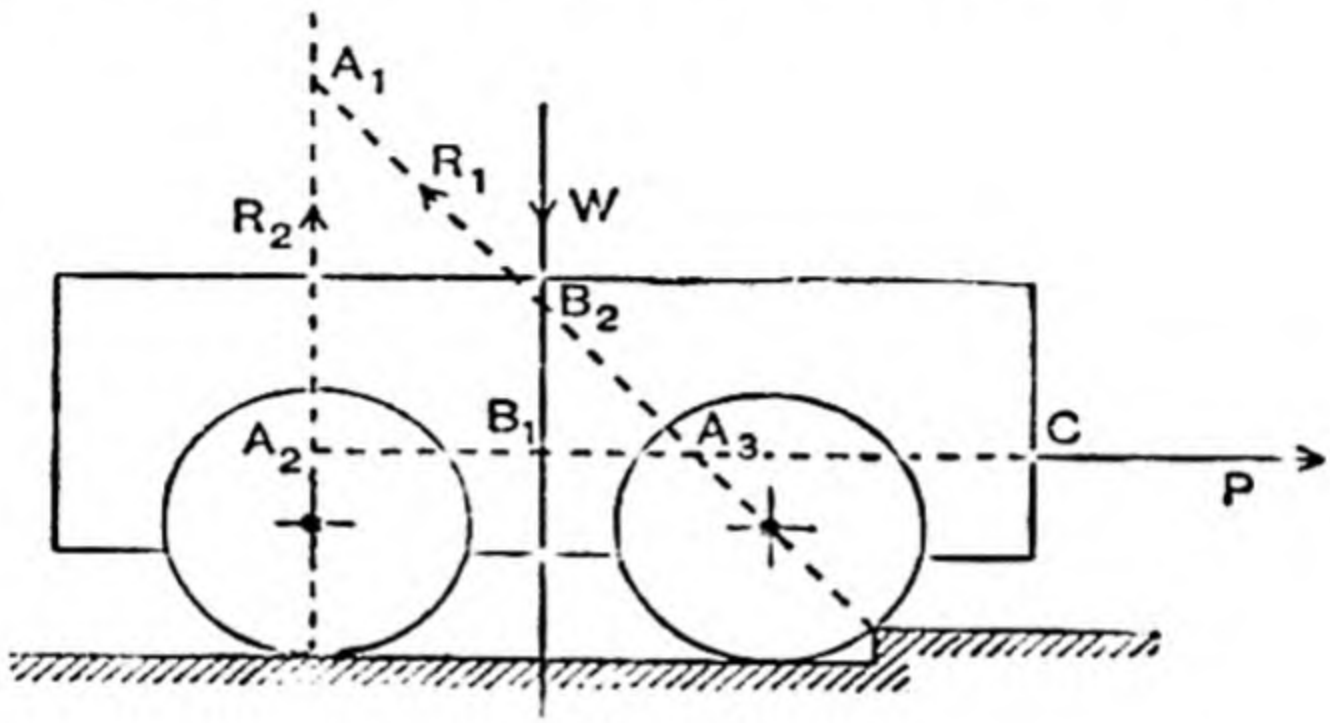

FIG. 15.

W and R_2 must also be vertical, so both vertical resultants must pass through A_3.

Using, instead of vector diagrams, moments of all the forces which keep the truck in equilibrium, we may choose A_1 as a point about which to equate moments, then

$$P \times A_1A_2 = W \times A_2B_1, \text{ which gives P,}$$

and about A_3, $R_2 \times A_2A_3 = W \times B_1A_3$, „ „ R_2.

And since P has no vertical component, the vertical component of R_1 is $W-R_2$, while its horizontal component is equal to P, and therefore

$$R_1 = \sqrt{\{(W-R_2)^2 + P^2\}}$$

which is easily calculated when R_2 and P have been found.

If P is not horizontal but passes through C, its moment about A_1 must still balance that of W about A_1; hence P

will be least when the arm or leverage of its moment about A_1 is greatest, *i.e.* when P acts perpendicular to the line from A_1 to C.

31. Funicular or Link Polygon. In determining fully certain incompletely specified forces to ensure that the three conditions of equilibrium are satisfied, we are concerned with the three items of magnitude, direction, and position, which go to make the complete specification of any force. To satisfy the conditions relating to magnitude and direction of forces in one plane, it is only necessary that the vector force polygon be closed (or that the sums of the horizontal and vertical components are each zero). But this does not completely satisfy the conditions of equilibrium, for the positions of the forces must be such that there is no resulting couple. In numerical calculations this is taken into account by so placing the equilibrating force or forces that the total moment of all the forces about one point in the plane is zero (see Example 2, Art. 27). This correct placing of the force or forces can alternatively be determined by graphical means which follow. We can replace any two forces by a single resultant through their point of intersection (see Art. 25), and continue the process to arrive at a single resultant or a couple which may be balanced by a single equilibrant or a couple as may be required. But this process fails for the important case of parallel forces, and is very inconvenient for forces which are nearly parallel, on account of the acute intersection of the lines of action, and the following alternative is then adopted.

It is convenient to use *Bow's Notation*, in which each force in the space or position diagram is denoted by two capital letters placed in the spaces on either side of the line of action and the same force in the vector diagram has the corresponding two small letters placed at its ends. Thus in Fig. 16, the force AB in the space diagram on the left corresponds to the vector *ab* in the polygon of forces on the right. To find the resultant or equilibrant of several forces such as P, Q, R, S, lettered AB, BC, CD, DE in Fig. 16, draw the vector force diagram *abcde* representing the forces in magnitude and direction by the length and direction of its sides. This gives the resultant *ae* (or the

equilibrant *ea*) in magnitude and direction, but not its position. The position is found by drawing a funicular or link polygon.

Any point, called a "pole" *o*, Fig. 16, is chosen in or about the vector polygon and joined to each vertex *a*, *b*, *c*, *d*, and *e*, and then from any point P, say on the line of action of AB, a line PT, called AO, is drawn parallel to *ao*, across the space A. From P a line (BO) drawn across the space B parallel to *bo* to meet the line of action of BC in Q; and from Q a line is drawn across the space C parallel to *oc* to meet CD in R. This process is continued until

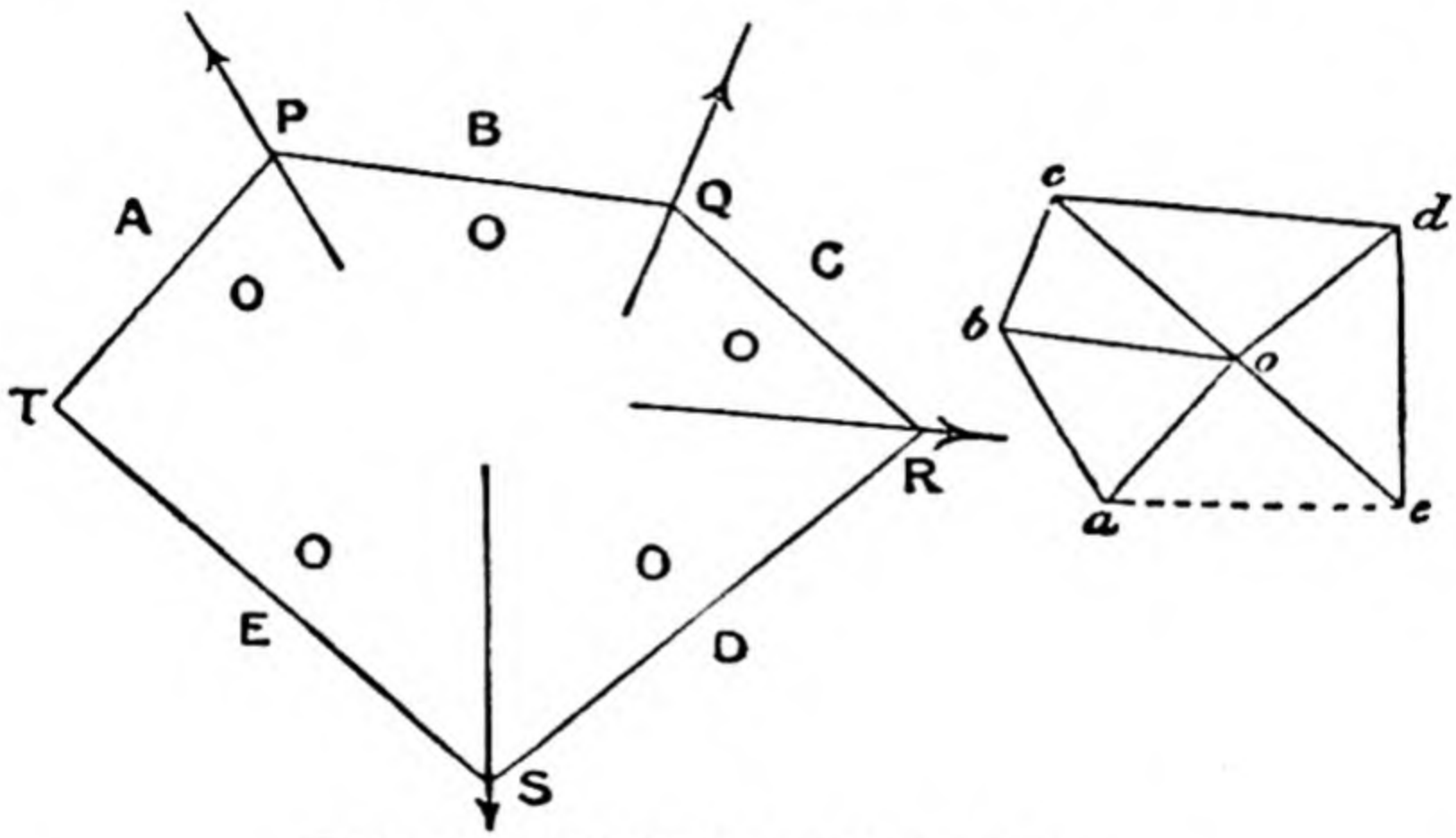

FIG. 16.—Funicular or link polygon.

finally a line EO is drawn from the intersection S parallel to *oe*, across the space E to meet the line AO in T. Then T is a point in the line of action of the resultant, the direction of which is given by *ae* in the vector diagram. Hence the equilibrant EA or the resultant AE is completely determined. The closed polygon PQRST, having its vertices on the lines of action of the forces, is called a *funicular* or *link polygon*. That T must be a point on the line of action of the resultant is evident from the following considerations. Any force may be resolved into two components along any two lines which intersect on its line of

action, for it is only necessary for the force to be the geometric sum of the components. Let each force, AB, BC, CD, and DE, be resolved along the two sides of the funicular polygon which meet on its line of action, viz AB along TP and QP, BC along PQ and RQ, and so on. The magnitude of the two components is given by the corresponding sides of the triangle of forces in the vector diagram, *e.g.* AB may be replaced by components in the lines AO and BO (or TP and QP), represented in magnitude by the lengths of the vectors *ao* and *ob* respectively. Similarly, CD is replaced by components in the lines CO and OD represented by *co* and *od* respectively. When this process is complete, all the forces AB, BC, CD, and DE are replaced by components, the lines of action of which are the sides TP, PQ, QR, etc., of the funicular polygon. Of these component forces, those in the line PQ or BO are represented by the vectors *ob* and *bo*, and therefore have a zero resultant. Similarly, all the other components balance in pairs, being equal and opposite in the same straight line, except those in the lines TP and TS, represented by *ao* and *oe* respectively. These two have a resultant represented by *ae*, which acts through the point of intersection T of their lines of action. Hence, finally, the resultant of the whole system acts through T, and is represented in magnitude and direction by the line *ae*; the equilibrant is equal and opposite in the same straight line.

Choice of Pole. In drawing the funicular polygon, the pole *o* (Fig. 16) was chosen in any arbitrary position, and the first side of the funicular polygon was drawn from *any* point P in the line AB. If the side AO had been drawn from any point in AB other than P, the funicular polygon would have been a similar and similarly situated figure to PQRST.

The choice of a different pole would give a different shaped funicular polygon, but the points in the line of action of the unknown equilibrant obtained from the use of different poles would all lie in a straight line. The choice of a suitable pole will generally lead to a well-shaped link polygon, *i.e.* one in which the intersections of successive sides are not at very acute angles. A badly chosen pole will give an ill-conditioned link polygon with intersections

so acute as to make the vertices difficult to locate exactly, or it may be, outside the limits of a sheet of drawing paper.

32. Graphical Conditions of Equilibrium. If we include the equilibrant EA (Fig. 16, Art. 31) with the other four forces, we have five coplanar forces in equilibrium, and (1) the forces or vector polygon *abcde* is closed; and (2) the funicular polygon PQRST is a closed figure. Further, if the force polygon is *not* closed, the system reduces to a single resultant, which may be found by the method just described (Art. 31).

It may happen that the force polygon is a closed figure,

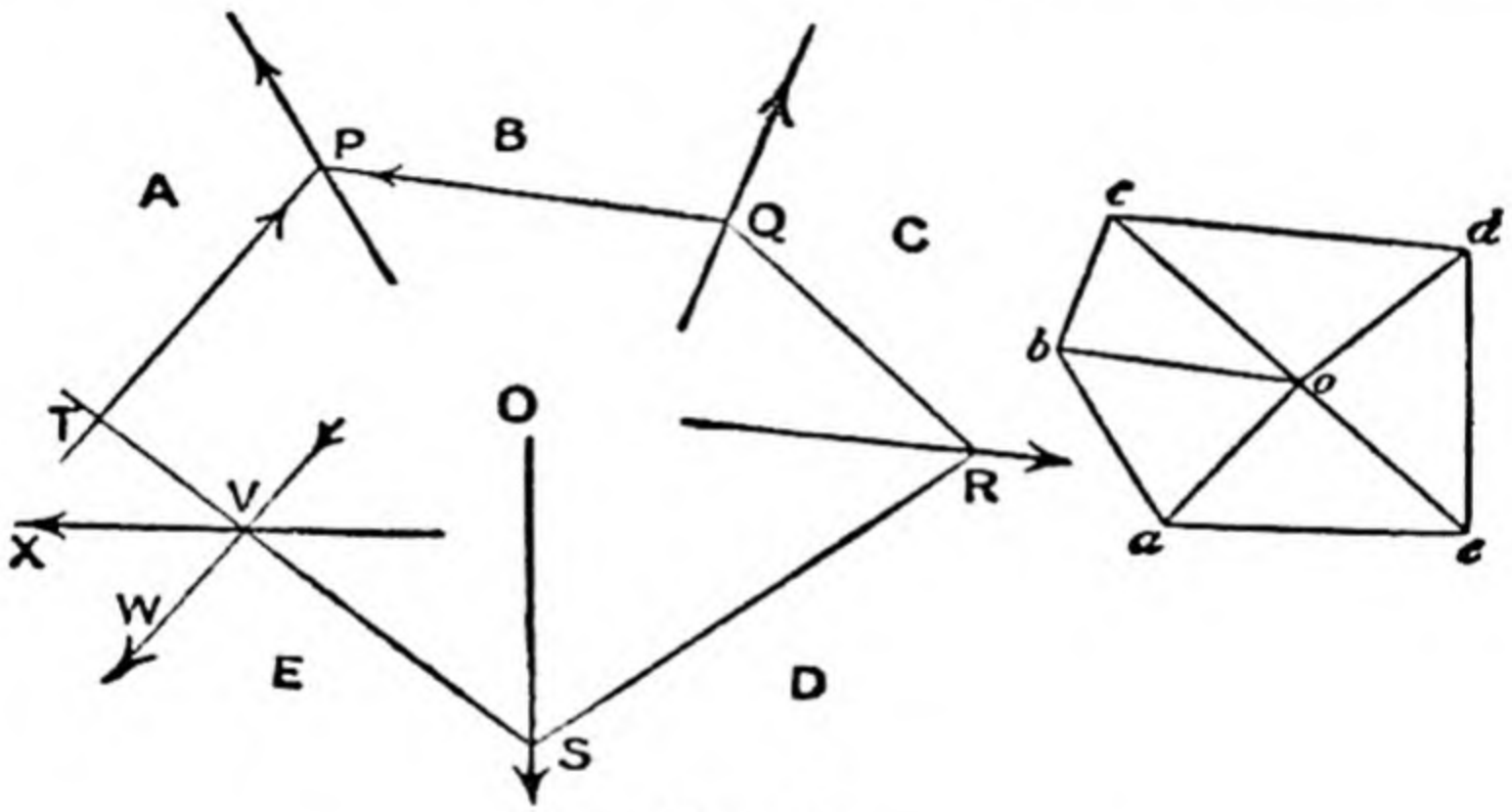

FIG. 17.

and that the funicular polygon is not. Take, for example, a diagram (Fig. 17) similar to the previous one, and let the forces of the system be AB, BC, CD, DE, and EA, the force EA *not* passing through the point T found in Fig. 16, but through a point V (Fig. 17), in the line TS. If we draw a line VW parallel to *oa* through V, it will *not* intersect the line TP parallel to *ao*, for TP and VW are then parallel. Replacing the original forces by components, the lines of action of which are in the sides of the funicular polygon, we are left with two parallel unbalanced components represented by *ao* and *oa* in the lines TP and VW respectively. These form a couple, and such a system is not in

equilibrium nor reducible to a single resultant. The magnitude of the couple is equal to the component represented by *oa* multiplied by the length represented by the perpendicular distance between the lines TP and VW. A little consideration will show that it is also equal to the force EA, represented by *ea*, multiplied by the distance represented by the perpendicular from T on the line VX. Or the resultant of the forces in the lines AB, BC, CD, and DE is a force represented by *ae* acting through the point T; this, with the force through V, and represented by *ea*, forms a couple.

Hence, for equilibrium it is essential that (1) the polygon of forces is a closed figure; (2) that the funicular polygon is a closed figure. These require the forces to satisfy three conditions of magnitude, direction, or position; and if of n non-concurrent coplanar forces in equilibrium (completely specified by $3n$ such conditions) $3n-3$ conditions are given, the remaining three can generally be found by the vector and link polygons. Thus, in Art. 31, the three conditions one magnitude, one direction, and one position, all relating to one equilibrant were determined; other cases having useful applications follow in Arts. 33 and 34.

33. Two and Three Equilibrants by the Link Polygon. It was shown in Art. 31 how to find by the link polygon a single equilibrant to a system of non-concurrent forces all in the same plane. The system may be balanced by two or by three equilibrants to comply with three conditions, and two important cases will now be explained.

(1) *Two Equilibrants.* Of n non-concurrent coplanar forces in equilibrium, given $n-2$ completely, one by its line of action (*i.e.* position and direction) and another by a point on its line of action (position), to find completely the n forces.

Let AB, BC, and CD (Fig. 18) be the lines of action of given forces represented in magnitude by *ab*, *bc*, and *cd* respectively in the vector polygon. Let ED be the line of action of one equilibrant, and p a point in the line of action of the second. Draw a line, *dx*, of indefinite length parallel to DE. Choose a pole, *o*, and draw in the funicular polygon corresponding to it, *but drawing first the side AO through the given point p*. Let the last side DO cut ED in *q*.

Since the complete funicular polygon is to be a closed figure, join *pq*. Then the vector *oe* is found by drawing a line, *oe*, through *o* parallel to *pq* to meet *dx* in *e*. The magnitude of the equilibrating force in the line DE is represented by the length *de*, and the magnitude and direction of the equilibrant EA through *p* is given by the length and direction of *ea*.

(2) *Three Equilibrants.* If three of the *n* forces are given by their lines of action, produce two of them to meet and

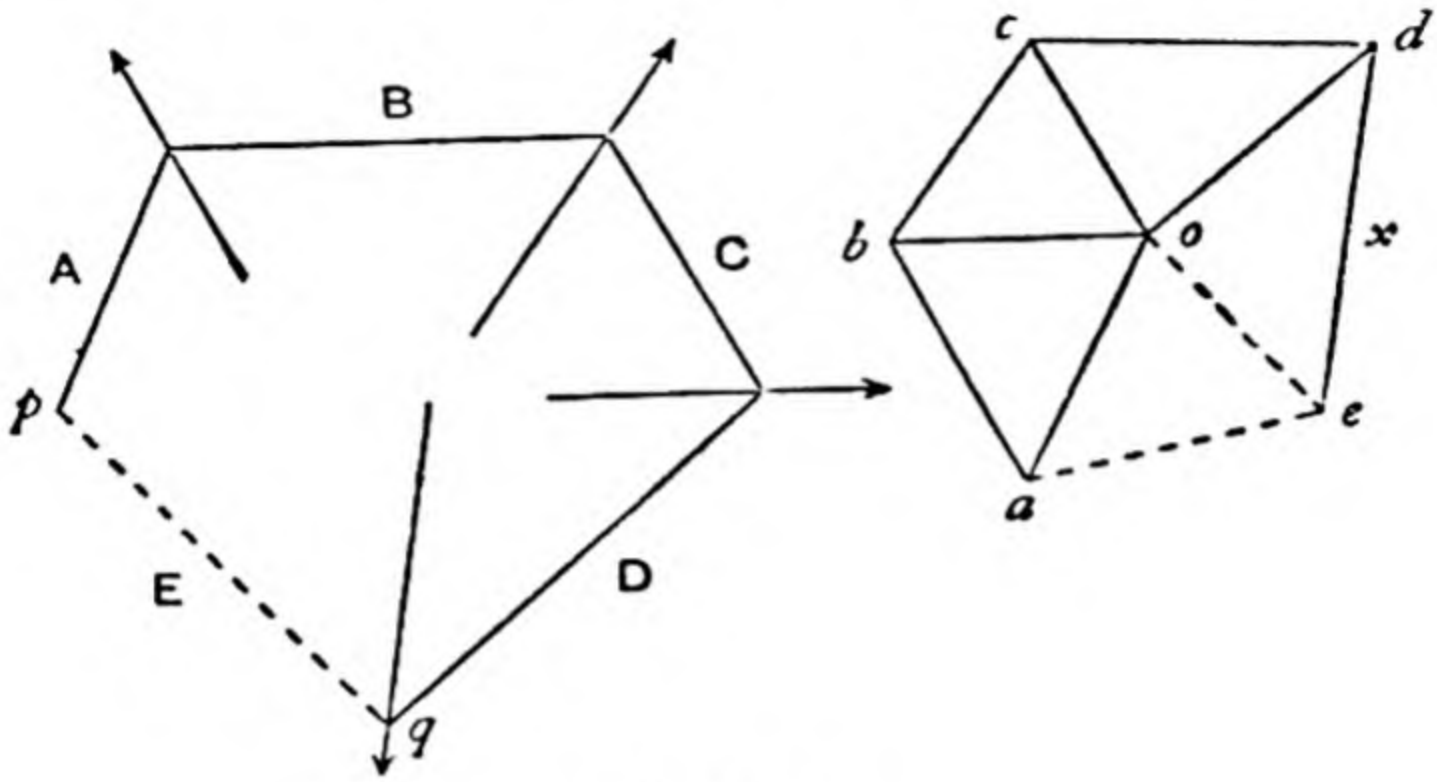

FIG. 18.

treat their intersection as the point *p* in Fig. 18, finally replacing the force through this point by two components along the lines which intersect there. The solution of this problem given in Art. 30 is generally more convenient than that of drawing a funicular polygon.

34. Funicular Polygon for Parallel Forces. To find the equilibrant or resultant of several parallel forces, the procedure is exactly the same as for non-parallel forces, but the vector polygon of forces has its sides all in the same straight line; it is "closed" if, after drawing the various consecutive vectors end to end, the last one terminates at the starting point of the first one.

In Fig. 19, four forces AB, BC, CD, DE are given and their resultant is required, *abcdc* is the open vector polygon, and

the magnitude and direction of the resultant is given by *ae*. A pole is chosen at *o* which is joined to *a*, *b*, *c*, *d*, and *e*. The

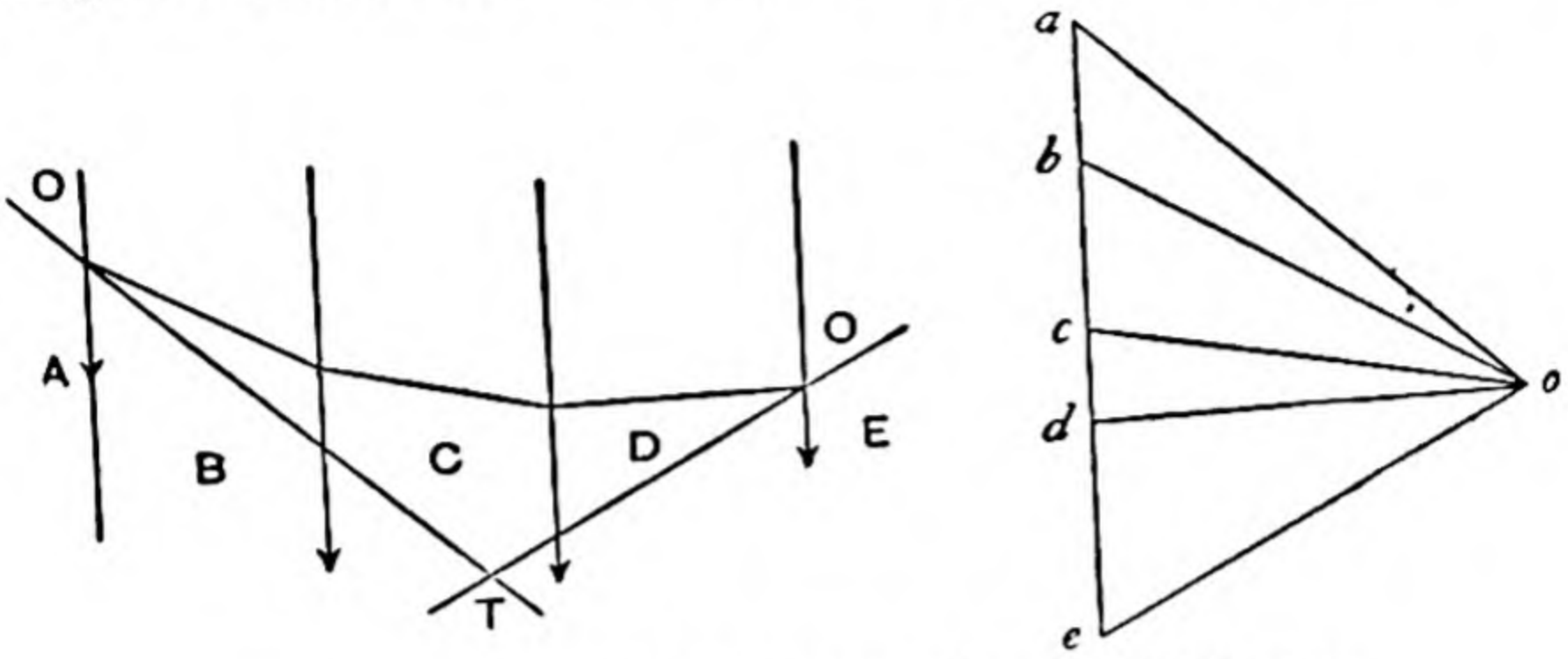

FIG. 19.—Funicular polygon for parallel forces.

funicular polygon having sides parallel to *ao*, *bo*, *co*, etc., is then drawn in the space diagram, starting from any arbitrary point. The extreme sides AO and EO (parallel to *ao* and *eo*) intersect in T, and this gives a point in the line of action of the resultant AE and so fixes its position. The equilibrant of the four forces is a force EA given in magnitude and direction by *ea* acting through T.

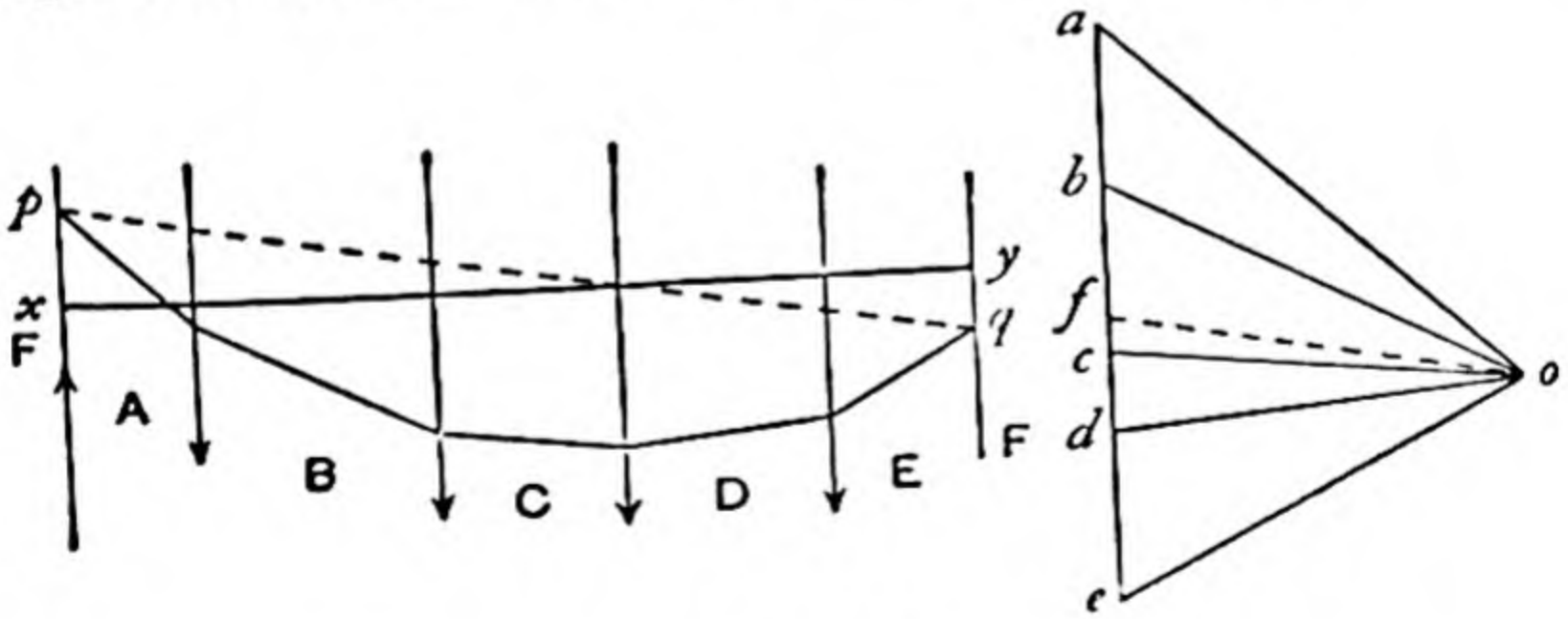

FIG. 20.

Two Equilibrants. If two parallel equilibrants to the given (vertical) forces AB, BC, CD, DE are required through two given points *x* and *y* (Fig. 20), choose a

pole, *o*, as before, and draw in the funicular polygon with side AO, BO, CO, DO, and EO respectively parallel to *ao*, *bo*, *co*, *do*, and *eo* in the vector diagram. Let AO meet the line FA (*i.e.* the vertical through *x*) in *p*, and let *q* be the point in which EO meets the line EF (*i.e.* the vertical through *y*). Join *pq*, and from *o*, draw a parallel line *of* to meet the line *abcde* in *f*. The magnitude of the upward equilibrant or supporting force in the line EF is represented by *cf*, and the other reaction in the line FA is represented by the vector *fa*. This may be proved in the same way as the proposition in Art. 31.

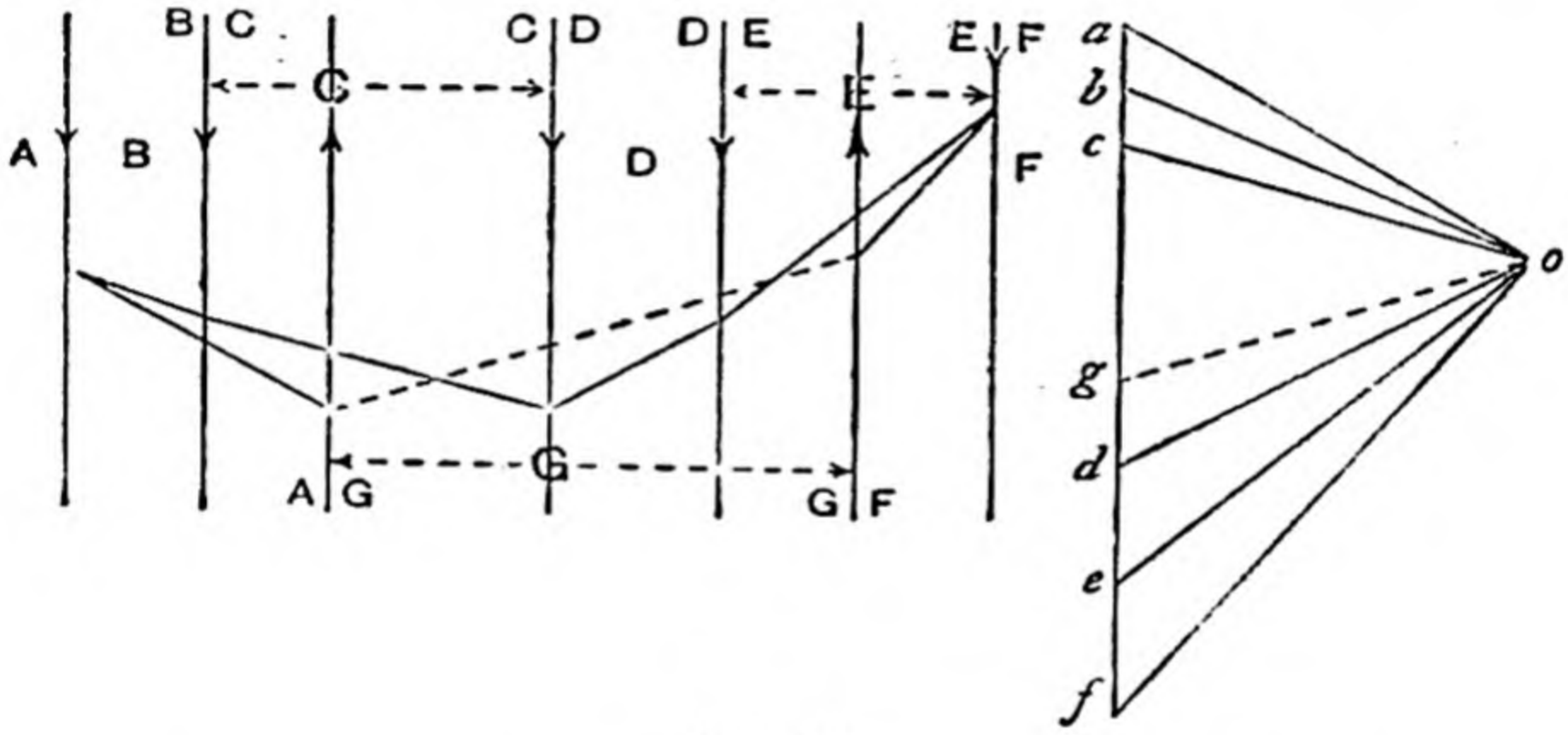

FIG. 21.

Another case is illustrated in Fig. 21 in which the two equilibrants FG and GA are not the extreme outside forces of the system; this presents no difficulty if the forces *ab*, *bc*, *cd*, *de*, *ef* are set off continuously on the vector diagram and the spaces lettered accordingly.

Thus the spaces C, E, and G extend as shown, from the lines BC to CD, DE to EF, and FG to GA respectively.

35. Moments of Parallel Forces from Funicular Polygon. The moments of parallel forces about a point in their plane can easily be found by means of the funicular polygon.

Let AB, BC, CD, DE, EF (or W_1, W_2, W_3, W_4, and W_5)

(Fig. 22) be five parallel (vertical) forces balanced by two equilibrants *fg* and *ga* (or R_2 and R_1). And let X be the point about which moments are required.

Let the funicular polygon for any pole *o*, starting, say, from *z*, be drawn as directed in Art. 34, *og* being drawn parallel to *zp* or GO, the closing line of the funicular, so that R_1, the left-hand equilibrant, is represented by the vector *ga* and R_2 by *fg*, while the loads W_1, W_2, W_3, W_4, and W_5 are represented by the vectors *ab*, *bc*, *cd*, *de*, and *ef* respectively. Consider any vertical line through X, at which the height of the link polygon is *xl*. Produce *xl* and the side *zw* to meet in *y*. Also produce the side *wm*

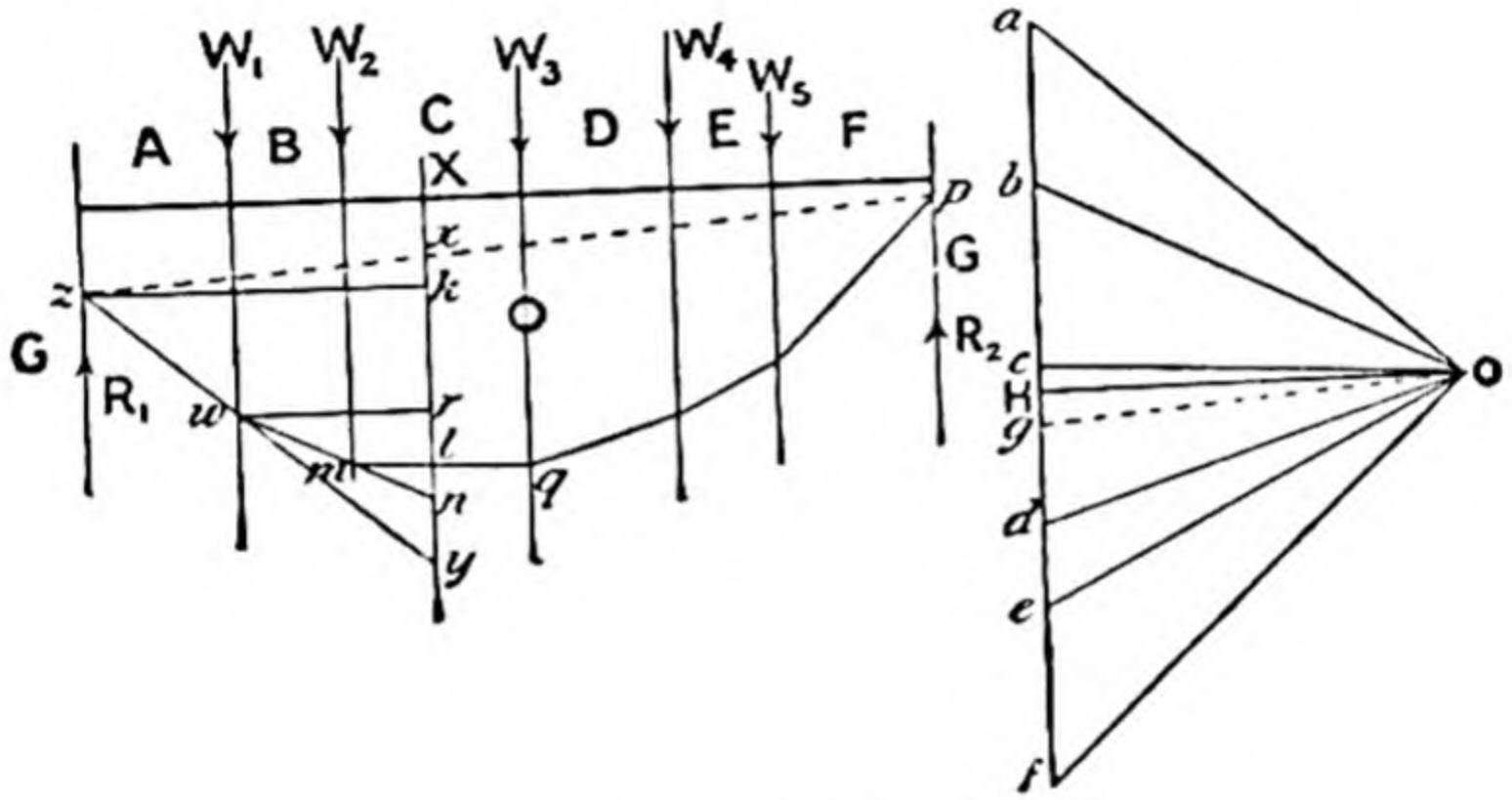

FIG. 22.

of the funicular polygon to meet *xy* in *n*, and let the next side *mq* of the funicular meet *xy* in *l*. The sides *zw*, *wm*, and *mq* (or AO, BO, and CO) are parallel to *ao*, *bo*, and *co* respectively. Draw a horizontal line, *zk*, through *z* to meet *xy* in *k*, a horizontal line through *w* to meet *xy* in *r*, and a horizontal *o*H through *o* in the vector polygon to meet the line *abcdef* in H. Then in the two triangles *xyz* and *gao*, there are three sides in one parallel respectively to three sides in the other; hence the triangles are similar, and

$$\frac{xy}{ag}=\frac{zk}{o\mathrm{H}},\quad \text{or}\quad xy\,.\,o\mathrm{H}=ag\times zk,\quad \text{or}\quad xy=\frac{ag\,.\,zk}{o\mathrm{H}}$$

Therefore, since ag is proportional to R_1, and zk is equal or proportional to the distance of the line of action of R_1 from X, $ag \,.\, zk$ is proportional to the moment of R_1 about X; and oH being an arbitrarily fixed constant, xy is proportional to the moment of R_1 about X.

Similarly

$$yn = \frac{ab \,.\, wr}{o\mathrm{H}}$$

and therefore represents the moment of W_1 about X to the same scale that xy represents the moment of R_1 about X. Hence xn or $xy - ny$ represents to the same scale the moment of the two forces R_1 and W_1 about X (or of their algebraic sum acting at their intersection of xz and wn). Similarly nl represents the moment of W_2 about X to the same scale, and

$$xl = xy - ny - ln$$

represents the moment about X of the three forces R_1, W_1 and W_2, or of their resultant (the algebraic sum) acting at the intersection of the lines xz and lm. For any point in the plane, and for any number of parallel forces, the proper intercept between the sides of the funicular polygon represents the moment and always to the same scale, since the distance from o to any side of the vector polygon $abcdefg$ is the same, viz. oH. For different pole distances (oH), the depth of closed link polygon will be inversely proportional to the pole distance.

Scales. If the scale of forces in the vector diagram is

1 inch to p lb.

and the scale of distance in the space diagram is

1 inch to q feet ;

and if oH is made h inches long, the scale to which the intercepts xl, xn, xy, ny, etc., represent the moments about X is

1 inch to $p \,.\, q \,.\, h$. lb.-feet.

30. Centre of Gravity, Centroid and Moment of Inertia. Evidently the funicular polygon can be used to find graphically the centre of gravity of a body if the centres of its constituent parts are all in one plane, for it lies in the

centre line of a number of parallel forces, *i.e.* in the line of action of their resultant. If the body has no line of symmetry, it may be necessary to find the resultants of parallel forces, representing the weights of parts of the body, in two directions to determine the c.g. by their intersection. This applies also to the c.g. of a thin lamina or to the centroid of a plane area. Similarly we can find the second moment or moment of inertia (Art. 9) of an area about any axis in its plane by the funicular polygon.

For regular shapes, simple algebraic expressions are obtainable to give the position of the c.g. or centroids and also the values of the moment of inertia I and radius of gyration k. For irregular shapes, division into parts and arithmetic tabulation of moments is often better than the funicular polygon. These calculations are essentially mathematics and we shall not burden these pages with them, but quote such as we may need.

The general formulæ (1) and (2) of Art. 8 are the foundations for finding the c.g. of regular bodies or areas or approximately, by tabulation, of irregularly shaped bodies or areas. We may notice that the c.g. of two small weights, w_1 at A and w_2 at B, divides AB inversely as the weights; or if G in Fig. 23 is the c.g. of the two weights

$$GA/GB = w_2/w_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{or} \quad GA = AB \times w_2/(w_1+w_2) \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{and} \quad GB = AB \times w_1/(w_1+w_2) \quad . \quad . \quad . \quad . \quad (3)$$

This gives the principle by which two partial masses can be replaced by a single one at their c.g., which is in effect replacing two forces by one resultant at the line of action at which the resultant has the same moment at all points in the plane of A and B as is due to w_1 at A and w_2 at B. This resultant (w_1+w_2) can similarly be combined with another weight w_3 and so on, the final c.g. of two weights being the c.g. of the original group of several weights.

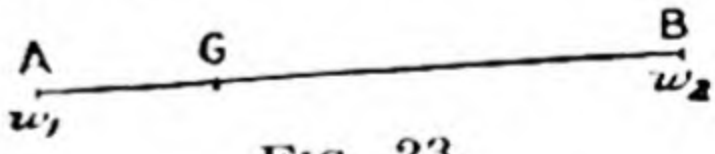

FIG. 23.

Solid or Area with Part Removed. To find the c.g. of a solid (or the centroid of an area) from which a part has been removed, we may treat the part removed and the remainder

as two masses the c.g. of which is at the c.g. of the whole body before removal of the part. Thus in Fig. 24 if the part B is removed and part A remains, let G be the c.g. of the complete body (A and B), G′ the required c.g. of the

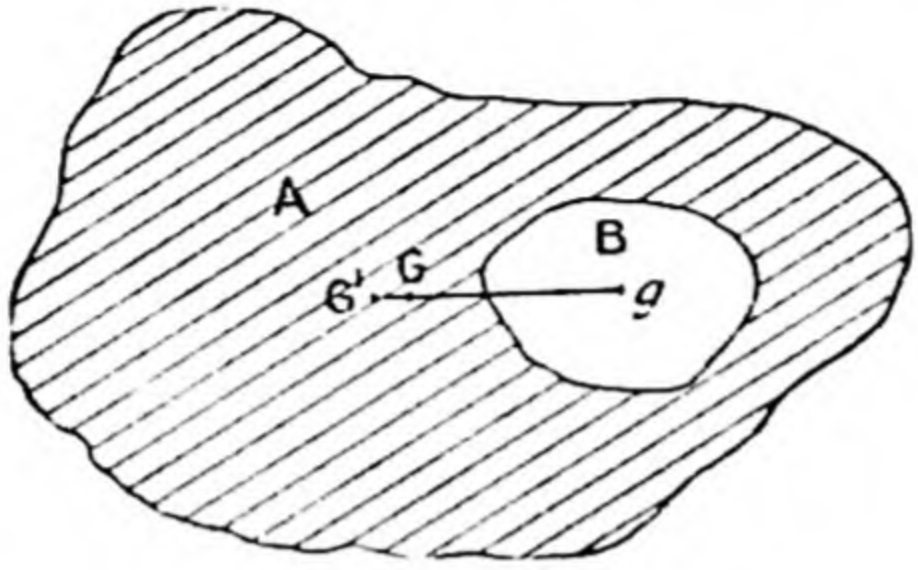

FIG. 24.

part A and g that of the removed part B. Then G divides G′g inversely as the masses of the two parts A and B, say, a and b respectively, or

$$GG'/Gg = b/a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$GG' = Gg \times b/a \quad . \quad . \quad . \quad . \quad . \quad (5)$$

EXAMPLE 1. A solid consists of a right circular cylinder 3 feet long and a right cone of height 2 feet, the base coinciding with one end of the cylinder, both parts being made of the same material. Find the c.g. of this solid, given that the c.g. of a cone is distant $\frac{1}{4}$ of the height from its base.

The solid is shown in Fig. 25. If a is the radius of the cylinder,

$$\text{vol. of cylinder/vol. of cone} = \pi a^2 \times 3/(\pi a^2 \times \tfrac{1}{3} \times 2) = 4{\cdot}5,$$

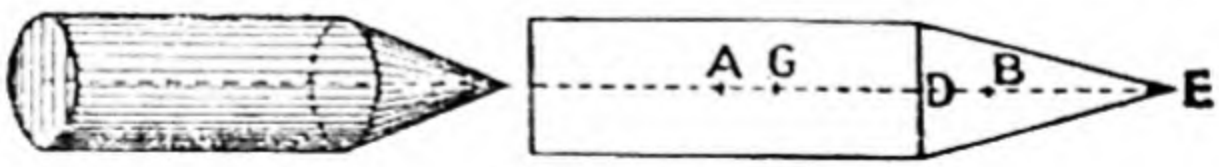

FIG. 25.

hence the weight of the cylinder is 4·5 times that of the cone. The c.g. of the cylinder is at A, the mid-point of its

axis, and that of the cone is at B, ½ of 2 feet or 0·5 feet from the common face at D, so that AB = AD + DB = 1·5 + 0·5 = 2 feet.

The distance from A of the c.g. G by (2), Art 36, is,

$$2 \text{ ft.} \times 1/(1+4{\cdot}5) = 2/5{\cdot}5 \text{ or } 4/11 \text{ ft.} = 4{\cdot}36 \text{ in.}$$

EXAMPLE 2. A pulley weighs 25 lb. but its c.g. is not accurately at its geometrical centre. It is balanced (*i.e.* the c.g. is brought to its geometrical centre) by drilling a small hole 6 in. from the geometrical centre, removing 0·1 lb. of metal. How far was the original c.g. from the centre of the pulley ?

Taking moments about the geometrical centre, if x is the required distance, a moment of 25 lb. $\times x$ is corrected by a reduction of 0·1 lb. $\times$ 6 in. = 0·6 lb.-in.

$$x = 0{\cdot}6 \text{ lb.-in.}/25 \text{ lb.} = 0{\cdot}024 \text{ in.}$$

EXAMPLE 3. An I-section is made up of three rectangles, viz. two flanges having their long sides horizontal, and one web connecting them having its long side vertical. The top flange section is 6 inches by 1 inch, and that of the

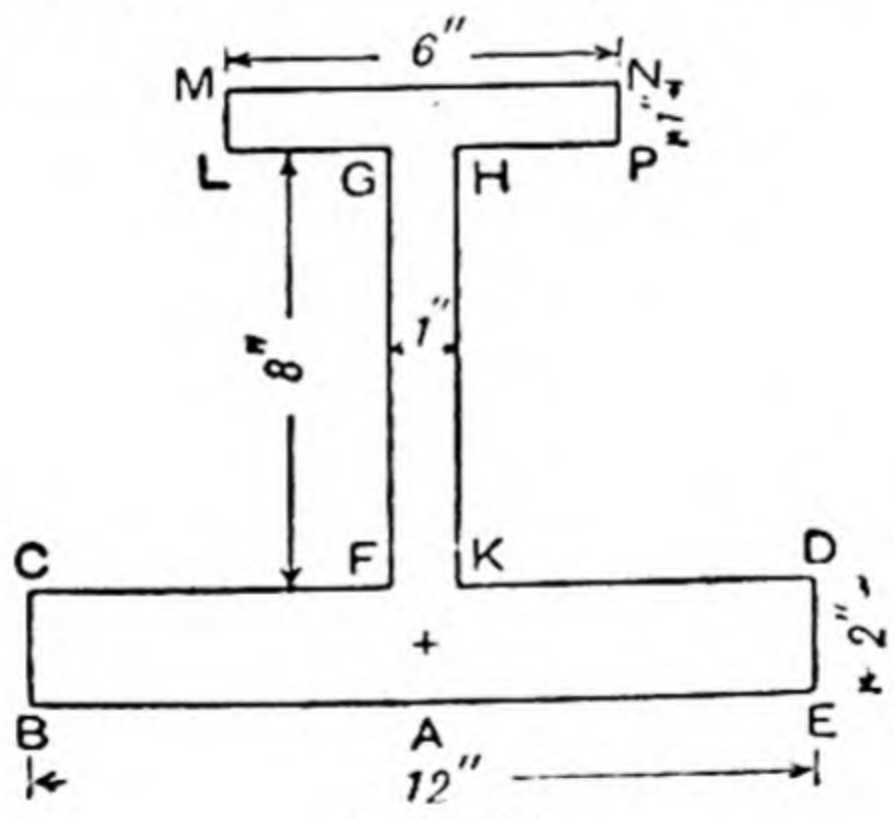

FIG. 26.

bottom flange is 12 inches by 2 inches. The web section is 8 inches deep and 1 inch broad. Find the height of the c.g. of the area of cross-section from the bottom of the lower flange.

Fig. 26 represents the section.

Let $\bar{x}$ = the required height above BE of the c.g. of the whole section.

The height of the c.g. of BCDE is 1 inch above BE ;

" " FGHK is $2+\frac{8}{2}=6$ inches above BE ;

" " LMNP is $2+8+\frac{1}{2}=10{\cdot}5$ inches above BE.

Equating the sum of the moments of these three *areas* about A to the moment of the whole figure about A, we have

$$(12\times2)1+(8\times1)6+(6\times1)10{\cdot}5=\bar{x}\{(12\times2)+(8\times1)+(6\times1)\}$$

$$24+48+63=\bar{x}(24+8+6)$$

$$\bar{x}=\frac{135}{38}=3{\cdot}55 \text{ inches.}$$

EXAMPLE 4. Find the c.g. of a cast-iron eccentric consisting of a short cylinder, 8 inches in diameter, having through it a cylindrical hole, 2·5 inches diameter, the axis of the hole being parallel to that of the eccentric and 2 inches

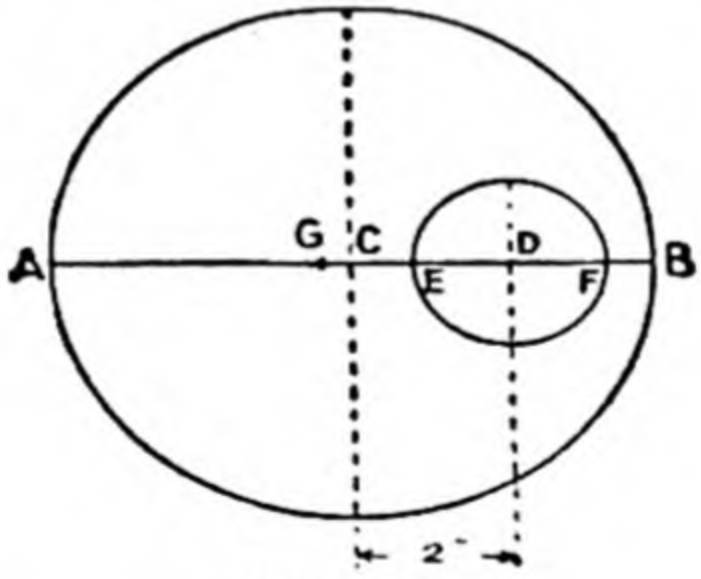

FIG. 27.

from it. State the distance of the c.g. of the eccentric from its centre.

This is equivalent to finding the c.g. of the area of a circular lamina with a circular hole through it. In Fig. 27,

AB = 8 inches. CD = 2 inches. EF = 2·5 inches.

Let the distance of the c.g. from A be $\bar{x}$.

If the hole were filled with the same material as the re-

mainder of the solid, the c.g. of the whole would be at C, its centre.

Equating moments of parts and the whole about A, omitting the common factor $\pi/4$,

$$\text{AC} \times (\text{area of circle AB}) = (\text{AD} \times \text{area of circle EF}) + (\bar{x} \times \text{area of eccentric})$$

$$4 \times 64 = 6 \times 6{\cdot}25 + \bar{x}(64 - 6{\cdot}25)$$

$$\bar{x} = \frac{256 - 37{\cdot}5}{57{\cdot}75} = 3{\cdot}783 \text{ inches}$$

Hence the distance of the c.g. from C $= 4 - 3{\cdot}783 = 0{\cdot}217$ inch.

37. Stability of Equilibrium. In order that a body may rest in equilibrium it is necessary that the resultant of the supporting forces should pass through the c.g., since this is the line of action of its weight or resultant of the force of gravity on its parts. For example, a cylinder (Fig. 28)

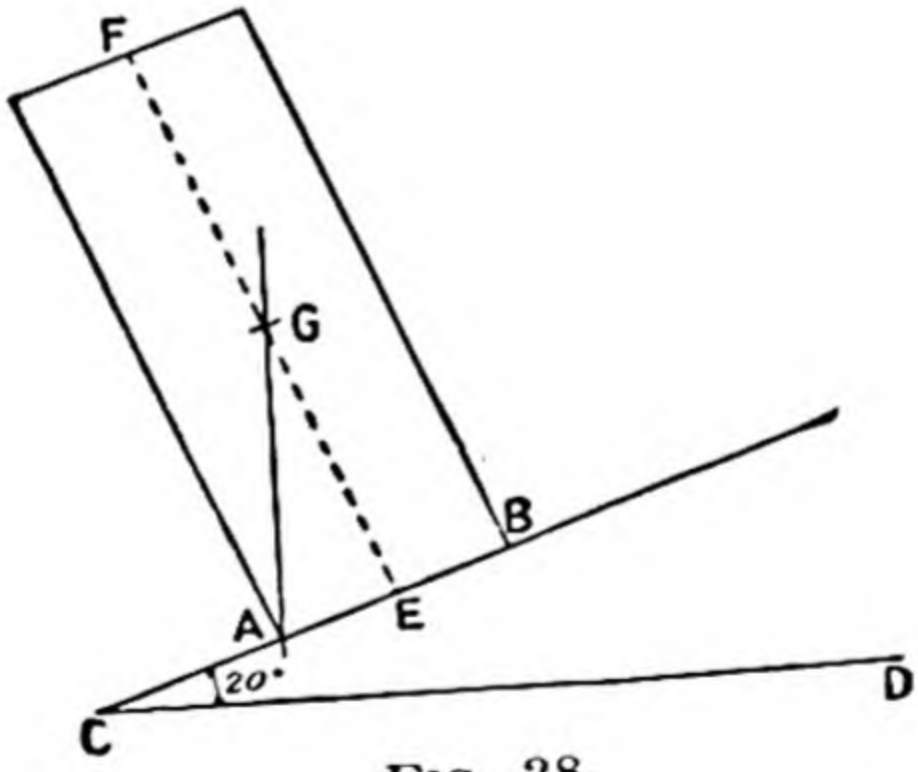

FIG. 28.

is shown resting on an inclined plane. So long as the c.g. at G is vertically above the base AB, the cylinder can stand. But if it were longer or if the angle of slope were greater it could not do so. In the case illustrated, on a slope of 20° to the horizontal the limit of height

$$\text{FE} = 2\text{GE} = 2(\text{radius of base} \times \cot 20°)$$

At this limit, a slight tilt to the left would cause the cylinder to fall over; its equilibrium is *unstable*. But if it were axially shorter or the slope were less steep, the resultant force of gravity through G would bring it back to place after a slight tilt, *i.e.* its equilibrium would be *stable*.

Stable, Unstable, and Neutral Equilibrium. A body is said to be in *stable* equilibrium when, if slightly disturbed from its position, the forces acting upon it tend to cause it to return to that position. If, on the other hand, the forces acting upon it after a slight displacement tend to make it go further from its former position, the equilibrium is said to be *unstable*. If, after a slight displacement, the forces acting upon the body form a system in equilibrium, the body tends neither to return to its former position nor to recede further from it, and the equilibrium is said to be *neutral*.

If a solid hemisphere ABN, Fig. 29, rests on a horizontal plane, and receives a small tilt, say through an angle θ,

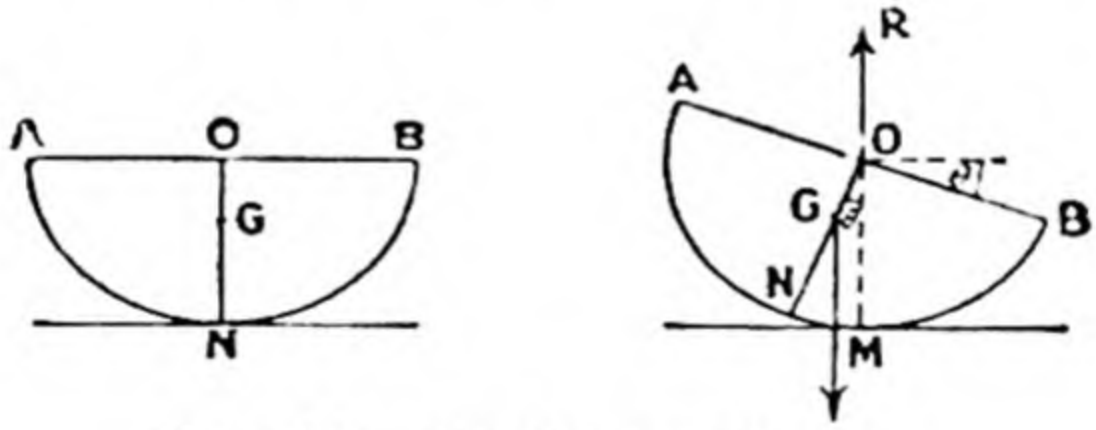

FIG. 29.—Stable equilibrium.

the c.g., situated at G, $\frac{3}{8}$ of ON from O and in the radius ON, takes up the position shown on the right hand of the figure. The forces acting instantaneously on the solid are then—(1) the weight vertically through G, and (2) the reaction R in the line MO vertically through M (the new point of contact between hemisphere and plane) and normal to the curved surface. These two forces form a "righting couple," and evidently tend to rotate the solid into its original position. Hence the position shown on the left is one of *stable* equilibrium. Note that G lies *below* O.

Suppose a solid consisting of, say, a cylinder with a hemispherical base, the whole being of homogeneous material, rests on a plane, and the c.g. G (Fig. 30) falls

within the cylinder, *i.e.* beyond the centre O of the hemispherical end, reckoned from N where the axis cuts the curved surface. On the left of Fig. 30, the solid is shown in a vertical position of equilibrium. Now suppose it to receive a slight angular displacement, as on the right side of the figure. The weight W, acting vertically downwards through G, along with the vertical reaction R of the plane, forms a couple, the tendency of which is to rotate the body so that G moves, not towards its former position, but away from it. The weight acting vertically through G and the reaction of the plane acting vertically through O form an "upsetting couple" instead of a "righting couple." Hence the position on the left of Fig. 30 is one of *unstable* equilibrium. Note that in this case G lies above O.

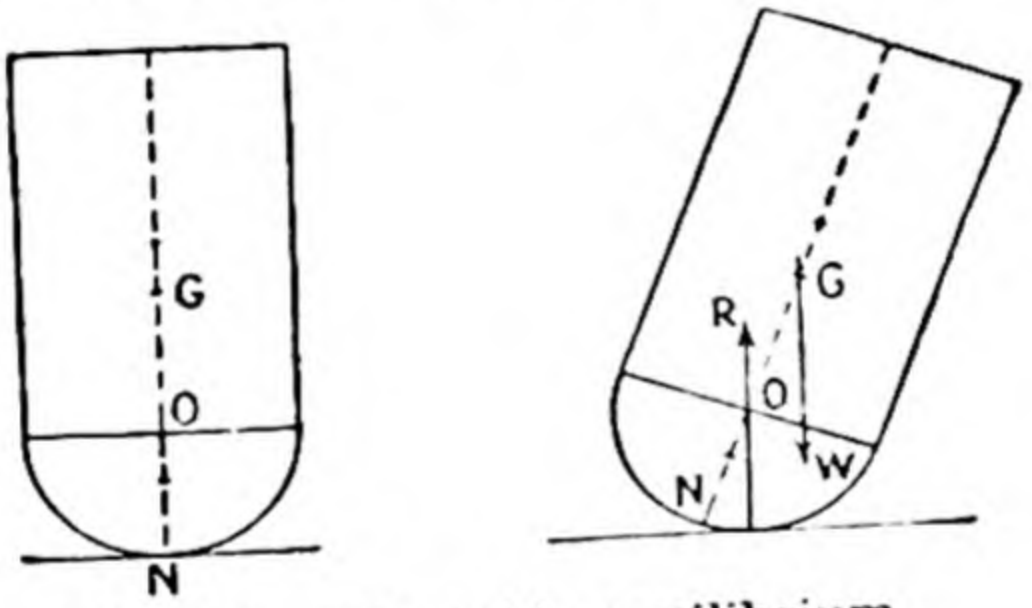

FIG. 30.—Unstable equilibrium.

If the upper part of the body were so small that G is below O, the equilibrium would be stable, as in the case of the hemisphere above. The lower G is, the greater is the righting couple (or the greater the stability) for a given angular disturbance of the body. While in the case of instability, the higher G is, the greater is the upsetting couple or the greater the instability, and we have seen that such a solid is stable or unstable according as G falls below or above O.

If G coincides with the centre of the hemisphere, the equilibrium is neither stable nor unstable, but neutral. Suppose the cylinder is shortened so that G, the c.g. of the whole solid, falls on O, the centre of the hemisphere.[1]

[1] For this, the moments of the hemisphere and cylinder about O must balance. If h is the length of cylinder of radius a

$$\tfrac{2}{3}\pi a^3 \times \tfrac{3}{8}a = \pi a^2 h \times \tfrac{1}{2}h, \text{ or } h = a/\sqrt{2}$$

Then if the solid receives a slight angular displacement, as in the right side of Fig. 31, the reaction R of the plane acts vertically upwards through O, the centre of the hemisphere (being normal to the surface at the point of contact), and the resultant force of gravity acts vertically downward through the same point. In this case the two vertical forces balance, and there is no couple formed and no tendency to rotate the body towards or away from its former position. Hence the equilibrium is neutral. Similarly, on a horizontal plane a sphere is in neutral equilibrium for angular displacement in any direction, while a cylinder is in neutral equilibrium for rolling displacements and in stable equilibrium for other angular displacements.

Stability of equilibrium in respect of angular displacements corresponds to minimum (gravitational) potential

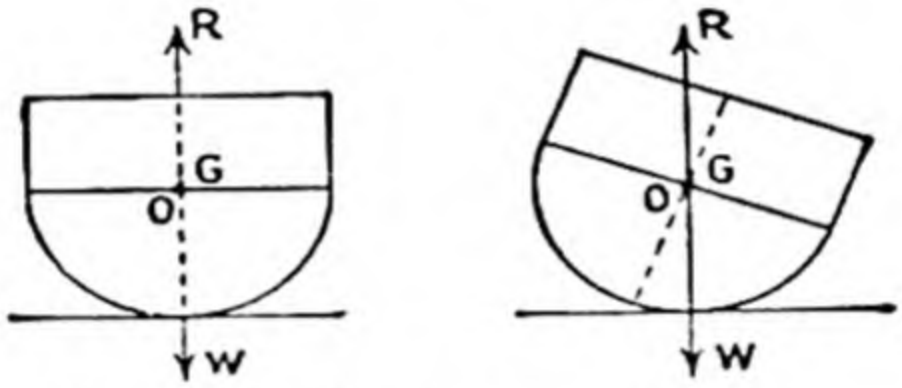

FIG. 31.—Neutral equilibrium.

energy. From the following article (38), it is evident that the potential energy depends on the height of the c.g. If a slight tilt raises the c.g. and so increases the potential energy, a body, if free to do so, will return to its position of minimum potential energy and low c.g., its increase of potential energy being drawn upon to set it in return motion. If the tilt lowers the c.g., thus decreasing the potential energy, the body has no increase of energy with which to restore its position, and, so far as the conditions are maintained, the displacement will increase. The position of instability is generally one of maximum potential energy. Finally, if the c.g. remains at the same level after disturbance, there is no change in potential energy with small angular displacement and the equilibrium is neutral.

38. Work Done in Lifting. When energy is spent in lifting a solid body or a liquid, it frequently happens that different portions are lifted through different distances, *e.g.* when a solid body is tilted, when water is raised from one container to another or when a hanging chain is wound up on an axle or drum. The total work done in lifting can be reckoned as follows: Let w_1, w_2, w_3, w_4, etc., be the weights of the various parts of the body, which is supposed divided into any number of parts, either large or small, but such that the whole of one part has exactly the same displacement. Let the parts w_1, w_2, w_3, etc., be at heights x_1, x_2, x_3, etc., respectively above some fixed horizontal plane. If the parts are not indefinitely small, the distances x_1, x_2, x_3, etc., refer to the heights of their centres of gravity. Then the distance $\bar{x}$ of the c.g. from the plane is $\Sigma(wx)/\Sigma(w)$, see (1) Art. 8. After the body has been lifted, let x_1', x_2', x_3', etc., be the respective heights above the fixed plane of the parts weighing w_1, w_2, w_3, etc. Then the distance $\bar{x}'$ of the c.g. above the plane is $\Sigma(wx')/\Sigma(w)$ (Art. 8).

The work done in moving the part weighing w_1 is equal to the weight w_1 multiplied by the distance $(x_1'-x_1)$ through which it is lifted; *i.e.* the work is $w_1(x_1'-x_1)$ units.

Similarly, the work done in lifting the part weighing w_2 is $w_2(x_2'-x_2)$. Hence the total work done is

$$w_1(x_1'-x_1)+w_2(x_2'-x_2)+w_3(x_3'-x_3)+, \text{ etc.}$$

which is equal to

$$(w_1x_1'+w_2x_2'+w_3x_3'+, \text{ etc.})-(w_1x_1+w_2x_2+w_3w_3+, \text{ etc.})$$

$$\text{or } \Sigma(wx')-\Sigma(wx)$$

$$\text{But } \Sigma(wx')=\bar{x}'\,\Sigma(w) \text{ and } \Sigma(wx)=\bar{x}\,\Sigma(w),$$

$$\text{therefore the work done} = \bar{x}'\,\Sigma(w)-\bar{x}\,\Sigma(w)$$

$$=(\bar{x}'-\bar{x})\,\Sigma(w)$$

The first factor, $\bar{x}'-\bar{x}$, is the distance through which the c.g. of the several weights has been raised, and the second factor, $\Sigma(w)$, is the total weight of all the parts. Hence the total work done in lifting a body is equal to the weight of the body multiplied by the vertical distance through which its c.g. has been raised.

EXAMPLE 1. A rectangular tank, 3 feet long, 2 feet wide, and 1·5 feet deep, is filled from a cylindrical tank of 24 square feet horizontal cross-sectional area. The level of water, before filling begins, stands 20 feet below the bottom of the rectangular tank. How much work is required to fill the tank, the weight of 1 cubic foot of water being 62·4 lb. ?

The water to be lifted is $3 \times 2 \times 1{\cdot}5$ or 9 cubic feet, hence the level in the lower tank will be lowered by $\frac{9}{24}$ or $\frac{3}{8}$ of a foot, *i.e.* by a length BC on Fig. 32. The 9 cubic feet of

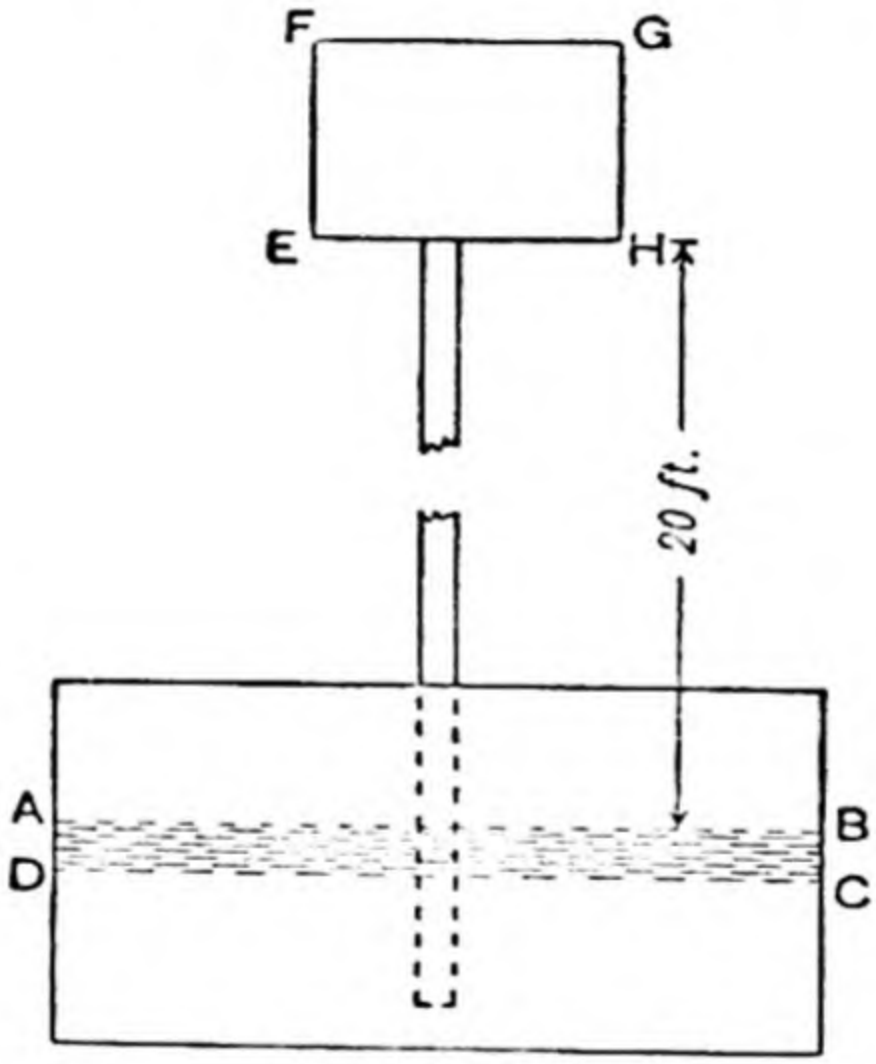

FIG. 32.

water lifted occupies first the position ABCD, and then fills the tank EFGH. In the former position its c.g. is $\frac{1}{2}$BC or $\frac{3}{16}$ foot below the level AB, and in the latter position its c.g. is $\frac{1}{2}$GH or $\frac{3}{4}$ foot above the level EH. Hence the c.g. is lifted $(\frac{3}{16}+20+\frac{3}{4})$ feet, *i.e.* $20\frac{15}{16}$ feet, or 20·9375 feet.

The weight of the 9 cubic feet of water lifted is $9 \times 62{\cdot}4 = 561{\cdot}6$ lb.

Hence the work done is $561{\cdot}6 \times 20{\cdot}9375 = 11{,}760$ foot-lb.

Example 2. Find the work in foot-pounds necessary to upset a solid right circular cylinder 3 feet diameter and 7 feet high, weighing half a ton, which is resting on one end on a horizontal plane.

Suppose the cylinder (Fig. 33) to turn about a point A on the circumference of the base. Then G, the c.g. of the cylinder, which was formerly 3·5 feet above the level of the horizontal plane, is raised to a position G′, *i.e.* to a height

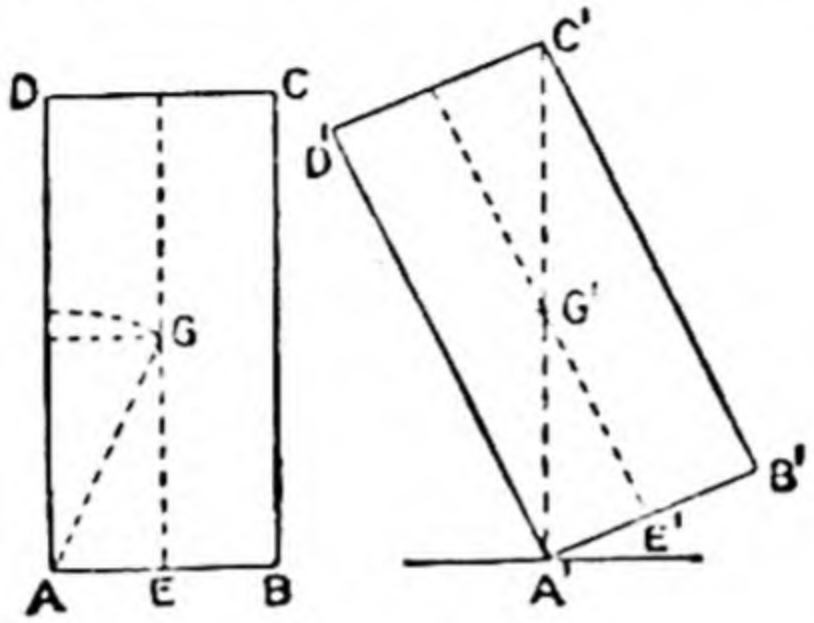

Fig. 33.

A′G′ above the horizontal plane before the cylinder is overthrown.

The c.g. is lifted a distance A′G′ −EG.

$$A'G' = \sqrt{(AE^2 + EG^2)} = \sqrt{(1{\cdot}5^2 + 3{\cdot}5^2)} = 3{\cdot}808 \text{ feet.}$$

The c.g. is lifted $3{\cdot}808 - 3{\cdot}5 = 0{\cdot}308$ foot,

and the work done is $1120 \times 0{\cdot}308 = 345$ foot-lb.

EXAMPLES II

1. A trap door, 3 feet square, is held at an inclination of 30° to (and above) the horizontal plane through its hinges by a cord attached to the middle of the side opposite the hinges. The other end of the cord, which is 5 feet long, is attached to a hook vertically above the middle point of the hinged side of the door. Find the tension in the cord, and the direction and magnitude of the pressure between the door and its hinges, the weight of the door being 50 lb., which may be taken as acting at the centre of the door.

2. A ladder, 20 feet long, rests on rough ground, leaning against a rough vertical wall, and makes an angle of 60° to the horizontal. The weight of the ladder is 60 lb., and this may be taken as acting at a point 9 feet from the lower end. The coefficient of

friction between the ladder and ground is 0·25. If the ladder is just about to slip downwards, find the coefficient of friction between it and the wall.

3. A ladder, the weight of which may be taken as acting at its centre, rests against a vertical wall with its lower end on the ground. The coefficient of friction between the ladder and the ground is $\frac{1}{3}$, and that between the ladder and the wall is $\frac{1}{4}$. What is the greatest angle to the vertical at which the ladder will rest?

4. A rod, 3 feet long, is hinged by a horizontal pin at one end, and supported on a horizontal roller at the other. A force of 20 lb., inclined 45° to the rod, acts upon it at a point 21 inches from the hinged end. Find the amount of the reactions on the rod at the hinge and at the free end.

5. A triangular roof-frame ABC has a horizontal span AC of 40 feet, and the angle at the apex B is 120°, AB and BC being of equal length. The roof is hinged at A, and simply supported on rollers at C. The loads it bears are as follows: (1) A force of 4000 lb. midway along and perpendicular to AB; (2) a vertical load of 1500 lb. at B; and (3) a vertical load of 1400 lb. midway between B and C. Find the reactions or supporting forces on the roof of A and C.

6. Draw a 2-inch square ABCD, and find the middle point E of AB. Forces of 17, 10, 8, 7, and 20 lb. act in the directions CB, AB, EC, ED, and BD respectively. Find the magnitude, direction, and position of the force required to balance these. Where does it cut the line AD, and what angle does it make with the direction AD?

7. A triangular roof-frame ABC has a span AC of 30 feet. AB is 15 feet, and BC is 24 feet. A force of 2 tons acts normally to AB at its middle point, and another force of 1 ton, perpendicular to AB, acts at B. There is also a vertical load of 5 tons acting downward at B. If the supporting force at A is a vertical one, find its magnitude and the magnitude and direction of the supporting force at C.

8. A jointed roof-frame, ABCDE, is shown in Fig. 34. AB and BC are inclined to the horizontal at 30°; EB and DB are in-

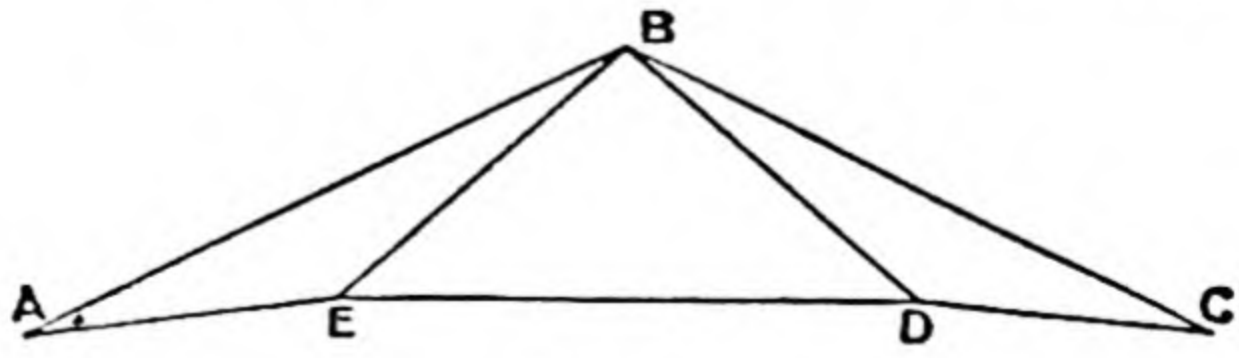

FIG. 34.

clined at 45° to the horizontal. The span AC is 40 feet, and B is 10 feet vertically above ED. Vertical downward loads of 2 tons

each are carried at B, at E, and at D. Find by the method of sections the forces in the members AB, EB, and ED.

9. A jointed structure, ACD . . . LMB (Fig. 35) is built up of bars, all of equal length, and carries loads of 7, 10, and 15 tons at

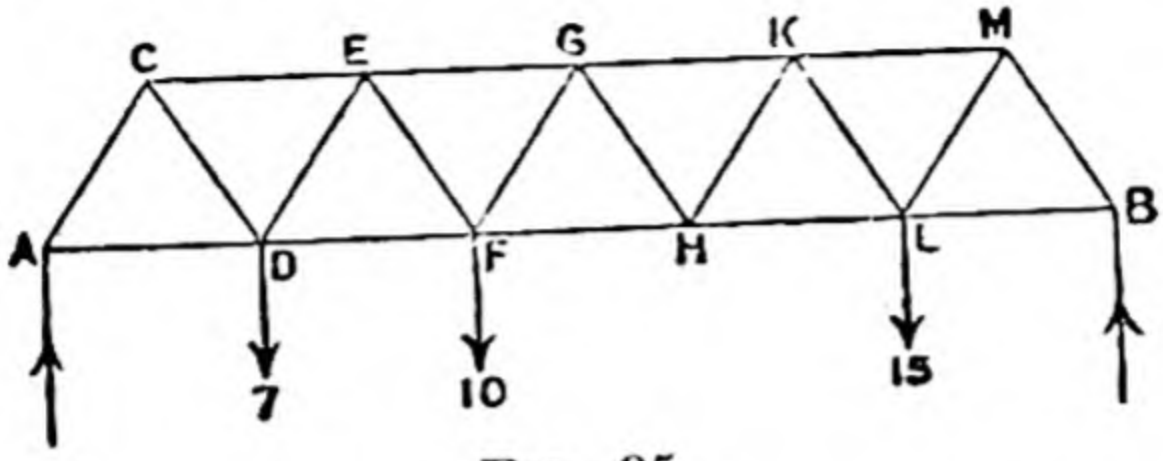

FIG. 35.

D, F and L respectively. Find by the method of sections the forces on the bars EF, EG, and DF.

10. Find the magnitude of the forces in the lines and directions FE, DA and AF to balance the forces given in Ex. No. 6, where F is the middle point of the line CD.

11. Solve Example 2 of Art. 30 for the following numerical data. Diameter of wheels 2 feet; distance apart of centres 3 feet; height of step 3 inches; level of pull P is 6 inches above wheel centres; load 5000 lb. midway between wheel centres.

12. Solve Example 2 of Art. 27 by drawing a link polygon.

13. A horizontal beam, 15 feet long, resting on supports at its ends, carries concentrated vertical loads of 7, 9, 5, and 8 tons at distances of 3, 8, 12, and 14 feet respectively from the left-hand support. Find graphically the reactions at the two supports.

14. A horizontal rod AB, 13 feet long, is supported by a horizontal hinge perpendicular to AB at A, and by a vertical upward force at B. Four forces of 8, 5, 12, and 17 lb. act upon the rod, their lines of action cutting AB at 1, 4, 8, and 12 feet respectively from A, their lines of action making angles of 70°, 90°, 120°, and 135° respectively with the direction AB, each estimated in a clockwise direction. Draw the funicular polygon to find the pressure exerted on the hinge, state its magnitude and its inclination to AB.

15. Find the c.g. of a T-girder section, the height over all being 8 inches, and the greatest width 6 inches, the metal being $\frac{7}{8}$ inch thick in the vertical web, and 1 inch thick in the horizontal flange.

16. A solid is made up of a hemisphere of iron of 3 inches radius, and a cylinder of aluminium 6 inches diameter, one end of which coincides with the plane circular face of the hemisphere. The density of iron being three times that of aluminium, what must be the length of the cylinder if the solid is to rest on a horizontal plane with any point of the hemispherical surface in contact?

17. A chain, weighing 12 lb. per foot and 70 feet long, hangs over a (frictionless) pulley, with one end 20 feet above the other. How much work is necessary to bring the lower end to within 2 feet of the level of the higher one ?

18. A hollow cylindrical boiler shell, 7 feet internal diameter and 25 feet long, is fixed with its axis horizontal. It has to be half filled with water from a reservoir, the level of which remains constantly 4 feet below the axis of the boiler. Find how much work is required to lift the water, its weight being 62·4 lb. per cubic foot.

19. A cubical block of stone of 3-feet edge rests with one face on the ground : the material weighs 150 lb. per cubic foot. How much work is required to tilt the block into a position of unstable equilibrium resting on one edge ?

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CHAPTER III

FRICTION AND MACHINES

39. Friction. The tangential or lateral force which resists the sliding motion of one body over another is called friction. It always acts against any tangential force tending to produce sliding motion, and work done against it transforms mechanical into heat energy. Though friction is actually complex, certain simple laws relating to friction are approximately true over a limited range of sliding speed and normal pressures between solids. According to these, the friction is independent of the speed of relative sliding motion, and of the area of surface in contact and proportional to the total pressure perpendicular to the sliding surface. The proportion which the force of sliding friction between two bodies bears to the normal pressure between them varies with the materials and is called the coefficient of friction and denoted by the symbol μ. Thus the friction F is given by

$$F = \mu Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where Q is the normal pressure between the two bodies when one is sliding over the other. A lateral pull insufficient to cause sliding is resisted by a smaller force equal and opposite to that lateral force. This smaller frictional force adjusts itself to the amount necessary to maintain equilibrium but has a limit beyond which it cannot rise and balance any applied lateral force P, and at this limit

$$F = P = \mu Q \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Actually, the limit of static friction is for many materials and conditions of surface somewhat higher than the quantity μQ where μ is the coefficient of sliding friction relating to the frictional force exerted after sliding motion has begun and is taking place. Or the limiting friction of rest is greater than the friction in motion.

Angle of Friction. Fig. 36 shows the forces acting on a body pulled laterally by a force P insufficient to cause

sliding motion. W is the weight of the body, Q the vertical reaction of the track, F the friction (actually in the plane of contact) and P an external force exerted on the body

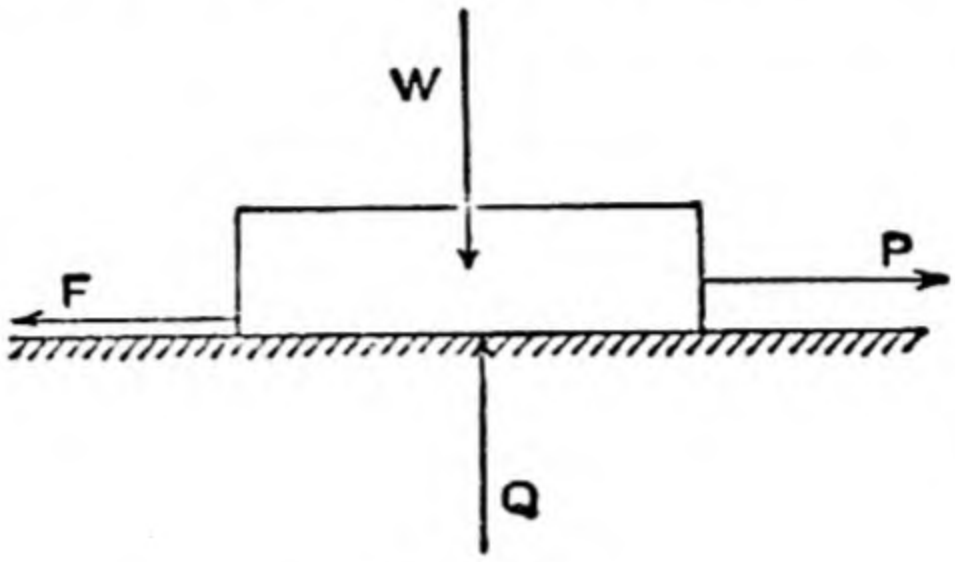

FIG. 36.

tending to make it slide on the track. Equating the horizontal and the vertical components, it is evident that Q = W and F = P, but unless P is in line with F, Q will not be in line with W. Fig. 37 shows F and Q combined

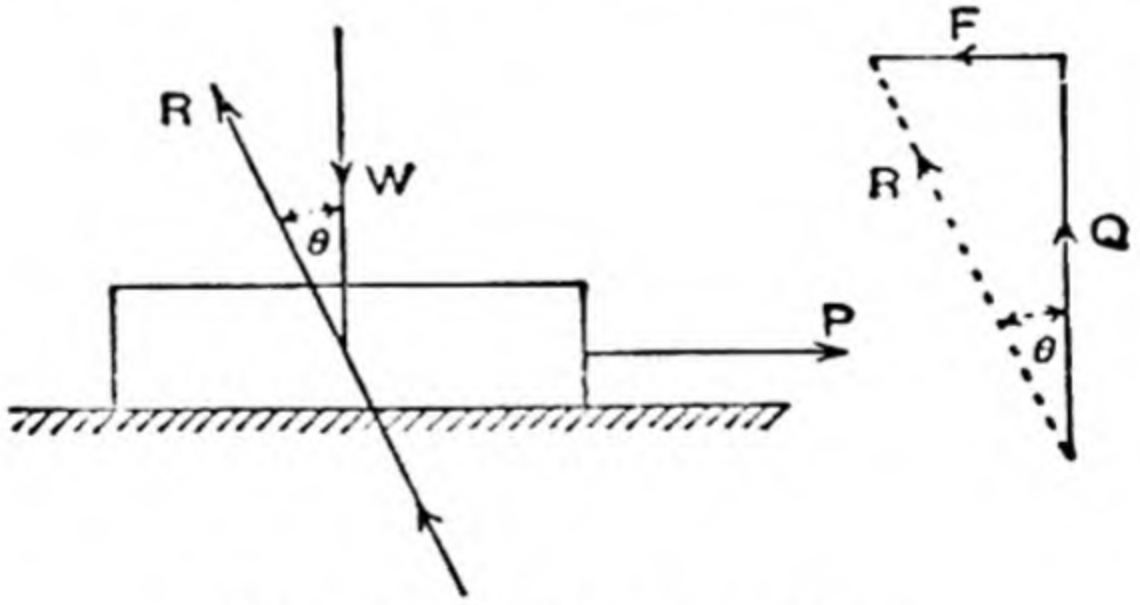

FIG. 37.—Angle of friction.

in a single resultant R, which with W and P (with both of which R must be concurrent) produce equilibrium. If θ is the inclination of R to the normal between the sliding surfaces, evidently $F = R \sin \theta = Q \tan \theta$. As F increases to resist an increasing external force P, θ increases; and when the limit of adjustment in F is reached, the limit in the magnitude of θ is called the *angle of friction*, ϕ, *i.e.* the greatest angle which the mutual forces between slider and

track can make with the common normal to the surface in contact. At this limit,

$$\tan \phi = \mu = F/Q \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where F is now the *limit* of the variable friction.

Body Resting freely on Incline. On a slope, α, less than the angle ϕ, a body can rest supported by friction F as shown at (*a*) of Fig. 38. The resultant force R between the body and the surface of the incline acts vertically and is equal to W the weight of the body. Its normal and tangential components are Q and F respectively, and $F/Q = \tan \alpha$. which is less than μ since α is less than ϕ (and $\tan \phi = \mu$). If α is increased to ϕ as shown at (*b*), Fig. 38, R is still just

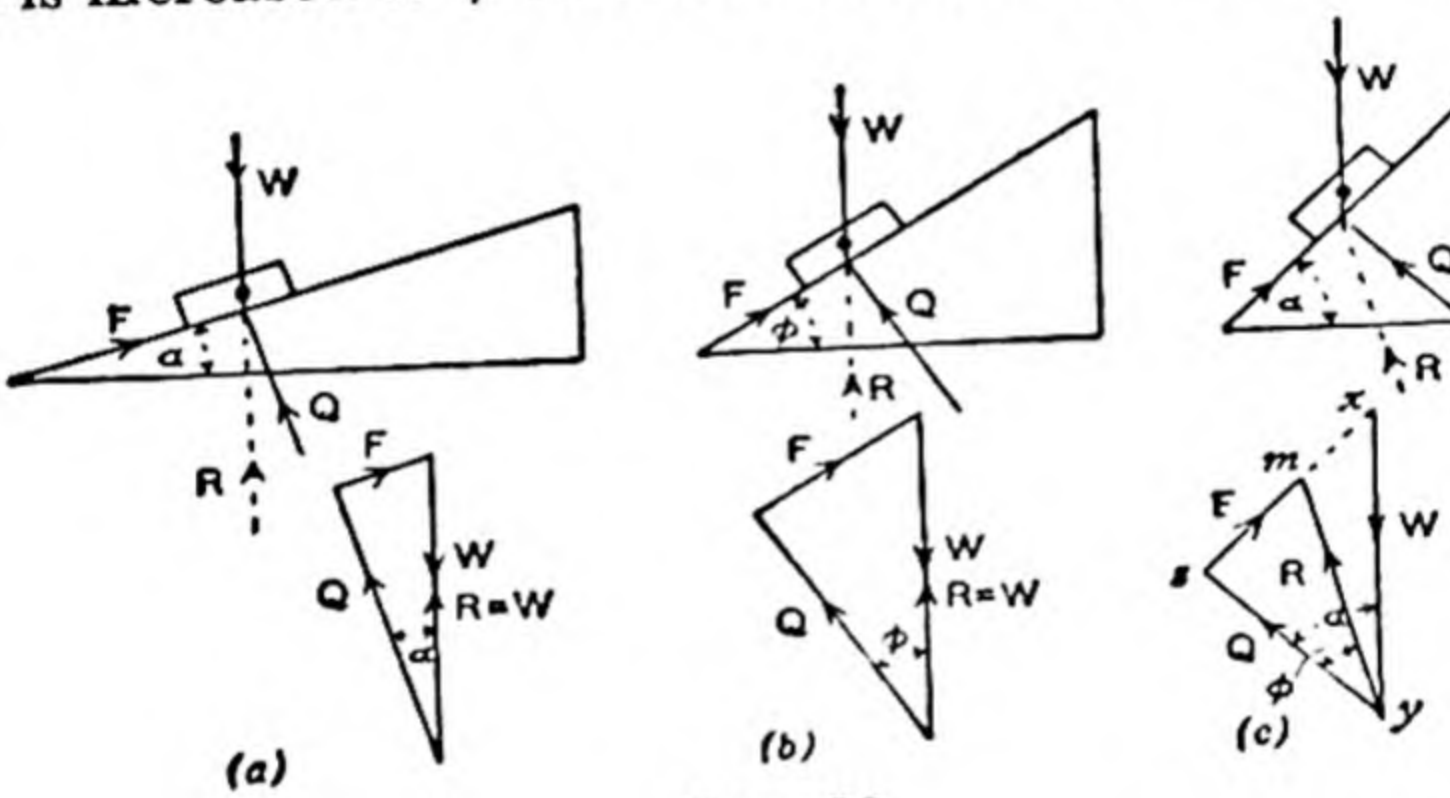

FIG. 38.

equal and opposite to W and in the same straight line, but $F/Q = \tan \phi = \mu$. If the slope, α, is still further increased beyond the value ϕ, the friction F or Q tan ϕ or μQ, represented by the vector zm, at (*c*), Fig. 38, is insufficient to balance the downward force W sin α or xz and motion down the slope takes place under the influence of the excess force xm.[1]

The critical angle ϕ is the greatest angle which the force between the body and the track can make with the normal

[1] With an acceleration

$$g \times \frac{W \sin \alpha}{W} - g \times \frac{F}{W} = g(\sin \alpha - \mu Q/W) = g(\sin \alpha - \mu \cos \alpha)$$

which is zero if $\alpha = \phi$, and $g \sin \alpha$ if $\mu = 0$.

to the track, and is also the greatest slope on which the body can rest freely on a track. It varies with the materials of the body and of the track and the condition of the surfaces.

40. Motion on Rough Incline. If P is a force at an angle θ to the inclined plane (*i.e.* at $\alpha+\theta$ to the horizontal) just sufficient to pull without acceleration a body of weight W up a track inclined α to the horizontal; and if ϕ is the angle of friction and μ $(=\tan\phi)$ is the coefficient of friction between the body and the track, the values of P and the force R exerted by the track on the body, are easily found from the trigonometry of the vector diagram of P, R, and W shown to the right of Fig. 39. For P is

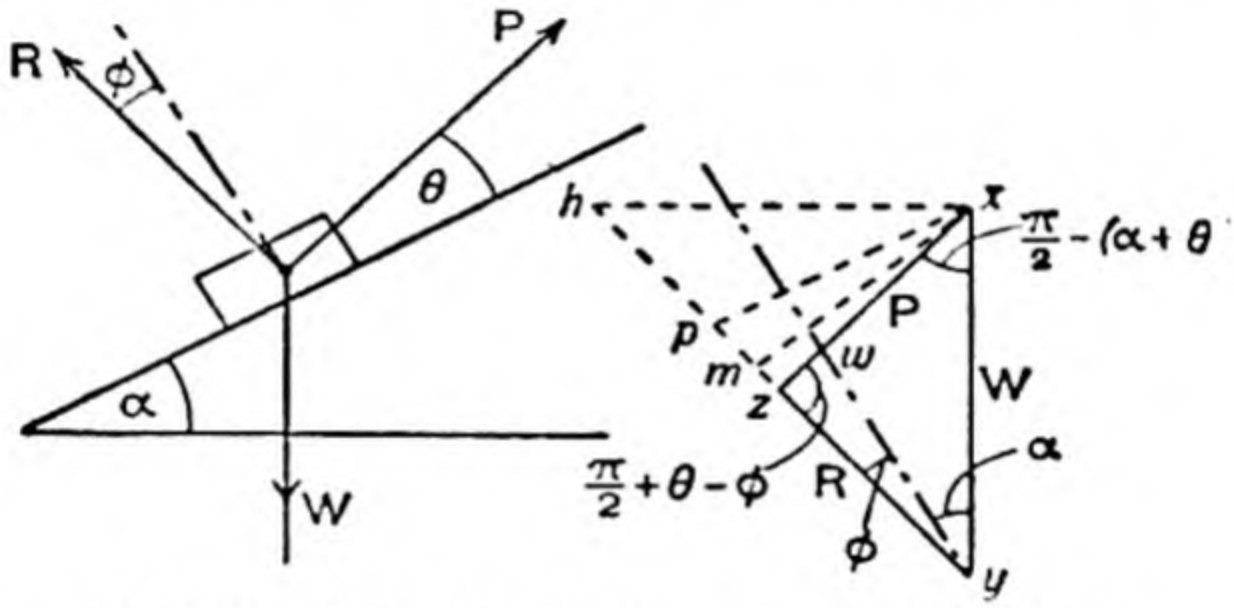

FIG. 39.—Steady motion up rough incline.

inclined $\theta+\alpha$ to the horizontal and $\frac{1}{2}\pi-(\theta+\alpha)$ to the vertical, or angle $z\hat{x}y=\frac{1}{2}\pi-(\theta+\alpha)$, yw is parallel to the normal to the sliding surface and therefore inclined α to the vertical, while R is inclined ϕ to the normal on the opposite side to P since its tangential component opposes motion up the slope. Hence R is inclined $\phi+\alpha$ to the vertical, or $x\hat{y}z=\phi+\alpha$. Hence $x\hat{z}y$ is $\frac{1}{2}\pi+\theta-\phi$

and
$$\frac{P}{\sin(\alpha+\phi)}=\frac{W}{\cos(\theta-\phi)}=\frac{R}{\cos(\theta+\alpha)} \quad . \quad . \quad (1)$$

$$P=W\sin(\alpha+\phi)/\cos(\theta-\phi) \quad . \quad . \quad . \quad (2)$$

and if $\theta=0$,

$$P=W\sin(\alpha+\phi)/\cos\phi \quad . \quad . \quad . \quad . \quad (3)$$

This is the pull parallel to the slope necessary to haul the body up the incline, and is obtained graphically by drawing from x in Fig. 39 a line parallel to the slope, as shown at xp, to meet yp in p. Since α and ϕ are fixed and only θ can be varied, the least pull P evidently from (2) will be when the denominator $\cos(\theta-\phi)$ is greatest, *i.e.* unity, or,

$$\cos(\theta-\phi)=1, \text{ so that } \theta-\phi=0, \text{ or } \theta=\phi \quad . \quad (4)$$

and then the pull will be

$$\text{minimum } P=W \sin(\alpha+\theta)=W(\sin\alpha+\phi) \quad . \quad (5)$$

Note that (2), (3), (4), and (5) hold good when $\alpha=0$, *i.e.* for a body sliding on a horizontal track.

This is graphically evident from Fig. 39, for the directions of W and R and the magnitude of W are fixed and the shortest line from x to complete a vector triangle is per-

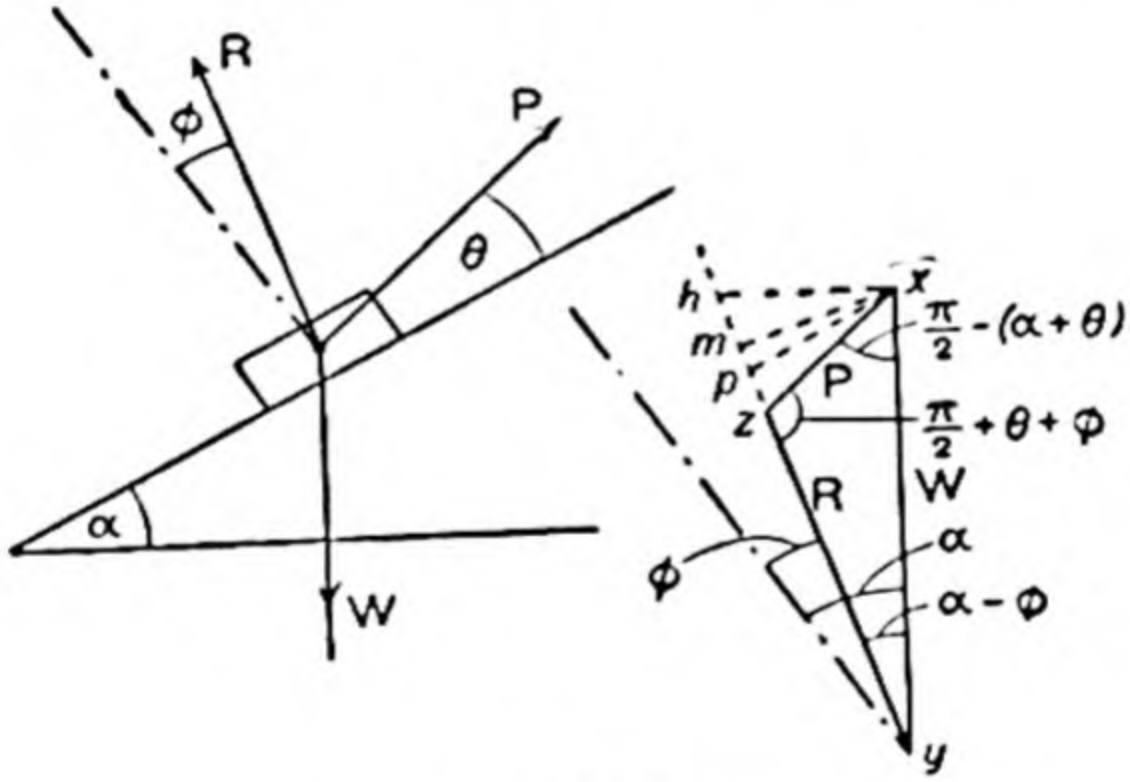

FIG. 40.

pendicular to R (as shown at xm). If P is horizontal, $\theta=-\alpha$, and

$$P=W \sin(\alpha+\phi)/\cos(\alpha+\phi)=W \tan(\alpha+\phi) \quad . \quad (6)$$

This is shown by hx, a horizontal line through x cutting yh in h.

The corresponding forces to allow the body to move *down* the slope without acceleration are easily found. The

friction now acts against motion and therefore up the incline. The vector diagram is shown at Fig. 40. Here

$$\frac{P}{\sin(\alpha-\phi)}=\frac{W}{\cos(\theta+\phi)}=\frac{R}{\cos(\theta+\alpha)} \quad . \quad . \quad (7)$$

The values of P for forces parallel to the slope, and horizontal, and the minimum value for any direction follow as before and are shown by the vector diagram of Fig. 40, with letters corresponding to those in Fig. 39.

Efficiency of Lift by Inclined Plane. The efficiency of a lifting device can be taken as the ratio of the useful work done in any lift to that expended by the effort, or is the ratio of the ideal effort which would be needed if there were no friction to the actual effort. Adopting the latter form, the efficiency for the rough inclined plane for any direction of the effort P is expressed by the ratio of the value of (2) when $\phi=0$ to the value of (2), as given in terms of α, θ, and ϕ. It will suffice to note the value for a horizontal effort P for which $\theta=-\alpha$; and from (6),

$$\text{Efficiency}=\tan\alpha/\tan(\alpha+\phi) \quad . \quad . \quad . \quad (8)$$

Slope for Maximum efficiency.

The efficiency (8) may also be written

$$\text{efficiency}=\frac{\sin\alpha}{\cos\alpha}\times\frac{\cos(\alpha+\phi)}{\sin(\alpha+\phi)} \text{ or } \frac{\sin(2\alpha+\phi)-\sin\phi}{\sin(2\alpha+\phi)+\sin\phi}. \quad (9)$$

which is greatest when $\sin(2\alpha+\phi)$ is greatest, *i.e.* when

$$2\alpha+\phi=90^\circ \text{ or } \alpha=45^\circ-\tfrac{1}{2}\phi,$$

which gives the slope of plane for greatest efficiency with a given angle of friction ($\tan^{-1}\mu$) or value of μ, and the efficiency (9) then becomes

$$\text{maximum efficiency}=\frac{1-\sin\phi}{1+\sin\phi} \quad . \quad (10)$$

41. Friction and Efficiency of Screw. A square-threaded screw is approximately the same in principle as an inclined plane when used with a nut to apply axial force by means of a relatively small effort applied perpendicular to the

axis and some distance from it, as in a screwjack used for lifting.

As shown in Fig. 41, the angle of the thread, α, is such that $\tan \alpha = p/\pi d$ where p is the pitch or lead of the screw, *i.e.* the distance it moves axially in a fixed nut per revolution of the screw, and d is the mean diameter of the screw, measured to the half depth of the thread. (Actually the angle of the thread varies a little from the bottom to the top of the thread since the travel is everywhere p per

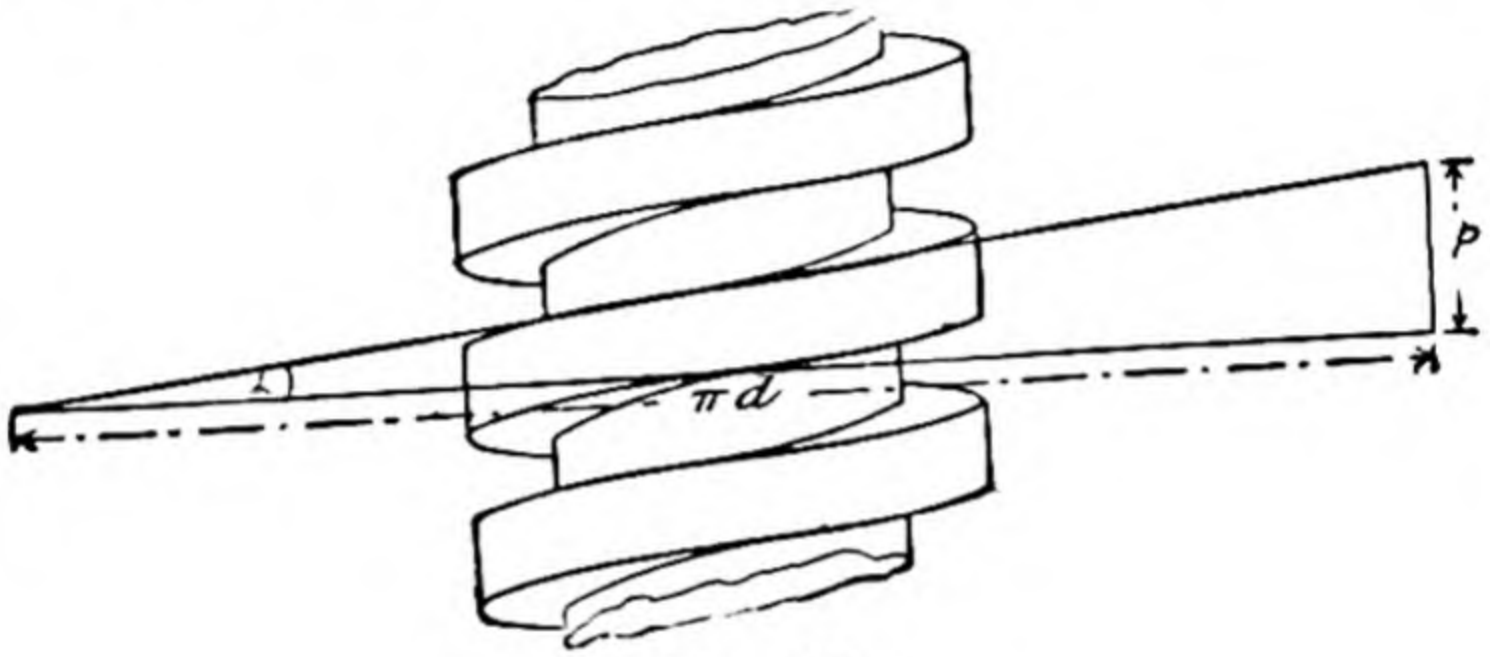

FIG. 41.

revolution, whereas the circumference of the thread of the screw is greater at the top of the thread than at the bottom.)

Regarding the screw as vertical and the horizontal effort P as applied tangentially at the mean radius of the screw, we have the relations as for the horizontal force P in Art. 40; and using the same notation, equation (6), Art. 40, for the screw becomes

$$P/W = \tan(\alpha+\phi) = \frac{\tan\alpha + \tan\phi}{1 - \tan\alpha \tan\phi} = \frac{\frac{p}{\pi d} + \mu}{1 - \frac{\mu p}{\pi d}} = \frac{p + \mu\pi d}{\pi d - \mu p} \qquad (1)$$

where $\mu = \tan\phi$, is the coefficient of friction; and for a frictionless screw, this would be $p/\pi d$. The efficiency, as in (8) of Art. 40, is

$$\tan\alpha/\tan(\alpha+\phi) \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

which can easily be put in terms of p, d, and μ.

For running downwards, as in (7) of Art. 40,

$$P/W = \sin(\alpha - \phi)/\cos(\theta + \phi) \quad . \quad . \quad . \quad . \quad (3)$$

which becomes zero for $\alpha = \phi$, and negative if α is less than ϕ, *i.e.* an effort would be required to lower the load. For values of α greater than ϕ, the screw requires a supporting effort to prevent it running down under the load. This condition begins when $\alpha = \phi$. Note that at that critical value the efficiency of *lifting* is, from (2)

$$\text{Efficiency} = \tan\alpha/\tan 2\phi = \tfrac{1}{2}(1 - \tan^2\phi) = 0{\cdot}5(1 - \mu^2) \quad (4)$$

an efficiency slightly *less than* 0·5, *e.g.* if $\mu = 0{\cdot}1$, the efficiency above which running down or reversal takes place is $\frac{1}{2}(1 - 0{\cdot}01) = 0{\cdot}495$.

It may also be noticed that while the use of a smaller pitch reduces α and the value of P as shown by (1) toward the limit W tan ϕ or μW when $\alpha = 0$, it also decreases the efficiency (2) towards the limit zero. Thus it is more useful by improved lubrication to reduce ϕ than to employ a very small pitch, for then much of the energy employed is wasted in friction.

42. Friction in Cylindrical Bearings. Friction Circle. When a cylindrical pin or shaft turns in a cylindrical hole (with the necessary small clearance), the mutual forces between the shaft and bearing will not be quite normal to the surface of the shaft, but inclined to the normal at the angle of friction. This is illustrated in Fig. 42, where the direction of rotation of the shaft is counter-clockwise. It will be observed that contact occurs at the point A, to which it rises from the lowest point in the bearing when motion takes place. R represents the reaction of the bearing on the shaft (R and the equal downward pressure of the shaft on the bearing are of course actually spread out in a line perpendicular to the figure or on what would approximate to a line in unyielding materials). It is vertical and equal in magnitude to W, the total load on

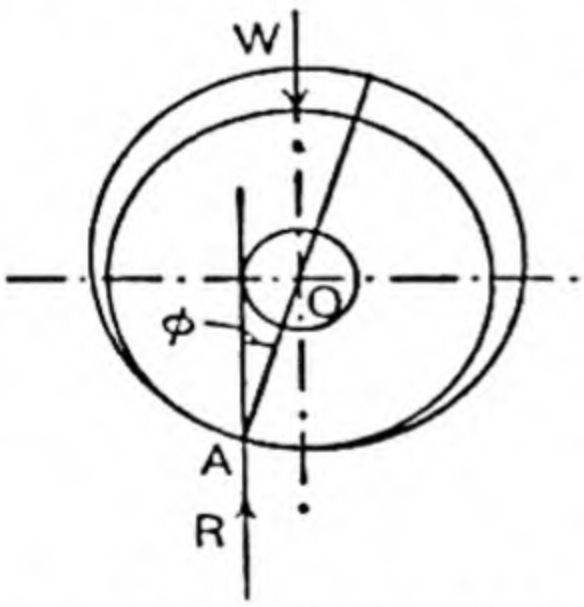

FIG. 42.—Friction circle.

the shaft, and inclined at an angle ϕ, the angle of friction, to the normal OA. It passes at a distance $r \sin \phi$ from O, where r is the radius of the shaft, and is tangent to a circle of radius $r \sin \phi$, which is called the *friction circle* for the combination of shaft and bearing. When relative motion is taking place between a pin and bearing, the thrust is tangential to the friction circle on one side or the other, according to the direction of motion. It follows that in a link with pins at both ends, the line of thrust in the link is not in the line of the centres of the pins but along the appropriate common tangent (of which there are four) to the friction circles of the two pins at the ends. If the pin is of small diameter in comparison with the length of the link or if the coefficient of friction is small, the deviation from axiality of the thrust is very small and often negligible. And as ϕ is generally small in a running shaft, $\sin \phi$ and $\tan \phi$ or μ may be considered as equal.

In Fig. 42 the (clockwise) frictional couple exerted on the shaft is

$$R \times r \sin \phi \text{ or } W \times r \sin \phi \quad . \quad . \quad . \quad . \quad (1)$$

and the work lost in friction, *i.e.* energy converted into heat per revolution, is $2\pi \times$ frictional couple, or

$$2\pi W r \sin \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which may be taken as equal to

$$2\pi \mu W r . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The ideal condition of lubrication of a bearing is that in which a continuous film of oil separates entirely the metal surfaces of the shaft and bearing. This is very difficult to attain and maintain. The opposite extreme condition is contact of metal with metal without lubricant, and this undesirable state too is scarcely attainable since traces of oil or grease are difficult to remove entirely. In most bearings, there is a partial separation of shaft and bearing by a film of oil. In comparison with the dimensions of oil molecules, the metal surfaces are rough; and on the high spots, there is some kind of bond between the metal and the molecules of the lubricant, and what is known as boundary lubrication exists. The conditions correspond rather to those of friction between two solids than to that between

a solid and a liquid film, but the liability to seizure is greatly reduced compared with that for chemically clean surfaces of metals.

43. Resistance to Rolling. It is well known that the resistance offered to rolling a cylinder, a ball, or a wheel is often much less than the resistance to sliding them along a similar track. The resistance to rolling a very hard wheel or ball on a very hard track is very small; but on softer or more yielding tracks, it is greater.

The resistance to rolling between two bodies is not perfectly understood, but it results from the indenting of the bodies by one another; there is a certain amount of sliding motion when the two bodies are in contact, due to the different degrees of strain in different parts of the indented areas of roller and track (or one body and the other). If both bodies are hard, the areas in contact are very small, but if one or both yield considerably there will be greater areas of contact and more relative sliding motion and greater resistance to rolling.

Ball and Roller Bearings. Advantage is taken of the small resistance to rolling in ball bearings, and also in

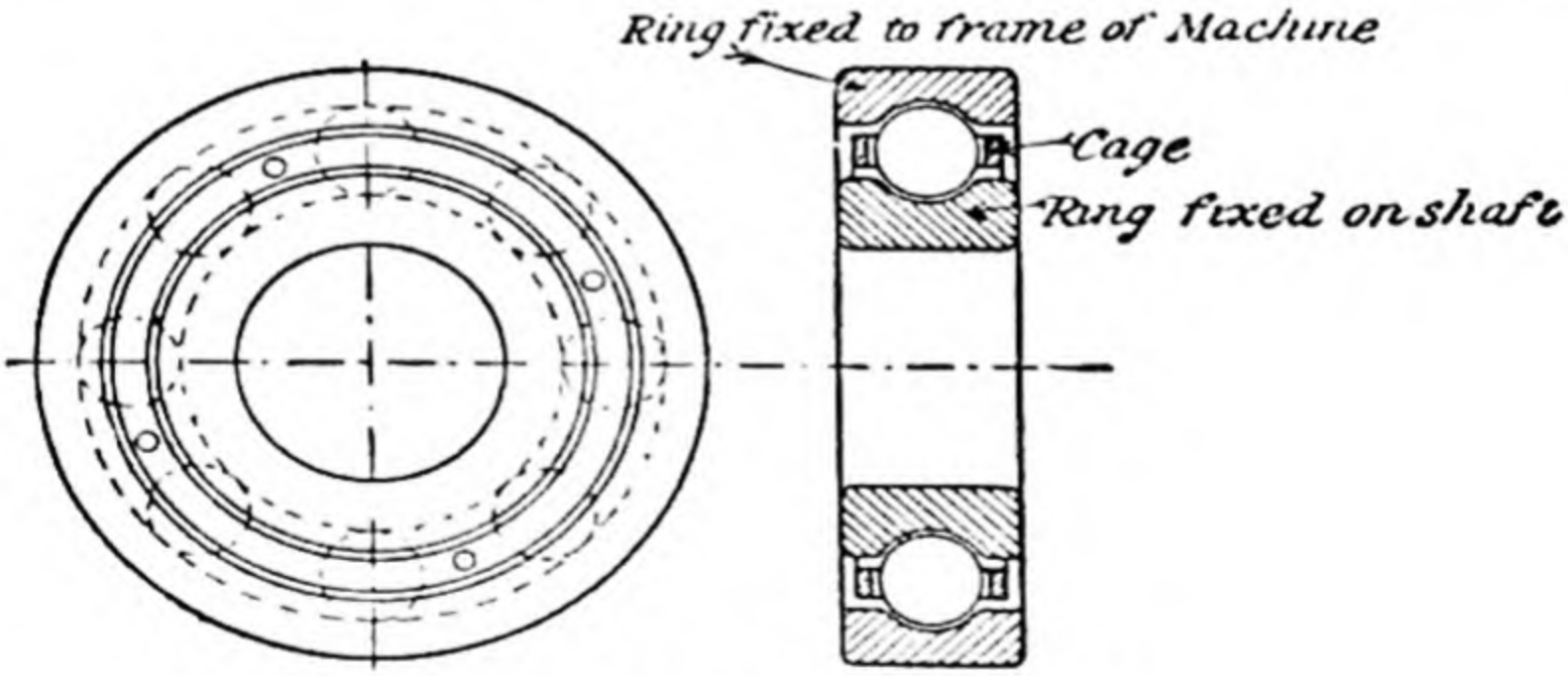

FIG. 43.—Journal ball bearing.

roller bearings; a very common example of the use of ball bearings occurs in bicycles, the free running of which is well known. Fig. 43 shows a ball journal bearing for carrying a rotating shaft which has no end thrust, while

Fig. 44 shows a single thrust ball bearing for carrying a shaft which has an end-thrust always in the same (downward) direction. In both cases the hard steel balls are carried in a gunmetal cage in which they are loosely

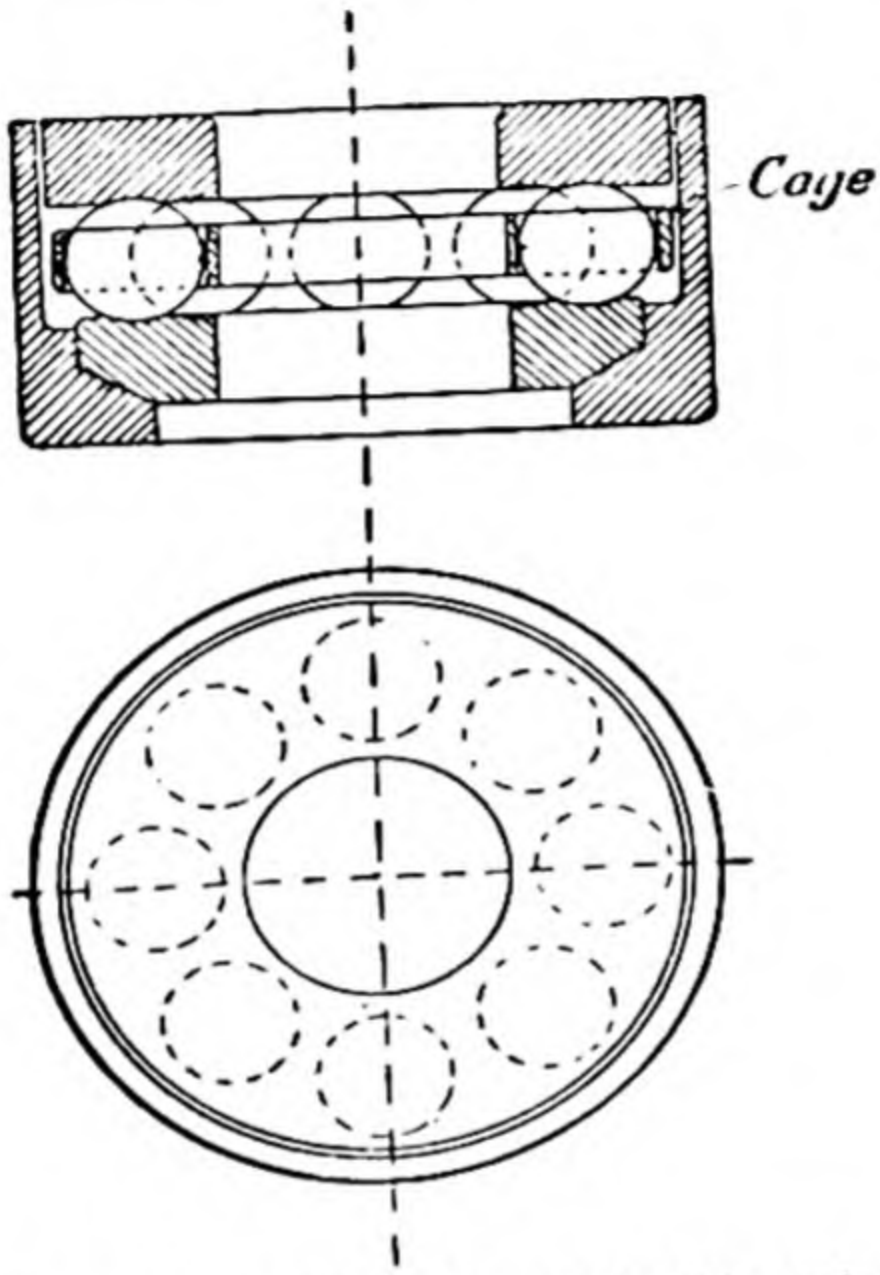

Fig. 44.—Single thrust ball bearing.

confined, kept at suitable distance apart and evenly distributed.

Fig. 45 shows a roller bearing used on the track wheels of many heavy waggons and agricultural machinery. The hard steel rollers are carried in gunmetal or malleable iron cages, and run between a hard steel sleeve fitted on the shaft and a hard steel bush fitting inside the outer casing, as shown. In both ball and roller bearings, oil serves to reduce wear by corrosion which is fostered by the varying stress to which the working parts are subject. Figs. 43, 44, and 45 are only selections of three very simple types of bearing from many in use, but they serve to show how in a bearing, rolling motion can be substituted for sliding motion.

Traction. The resistances experienced when propelling a wheeled vehicle on tracks of various kinds are usually stated in pounds per ton. For example, for a truck on steel rails an approximate value of the tractive resistance is 12 lb. per ton; but the value varies greatly with the speed; for low speeds mechanical friction is the most important, while at high speeds the air friction is greater than the

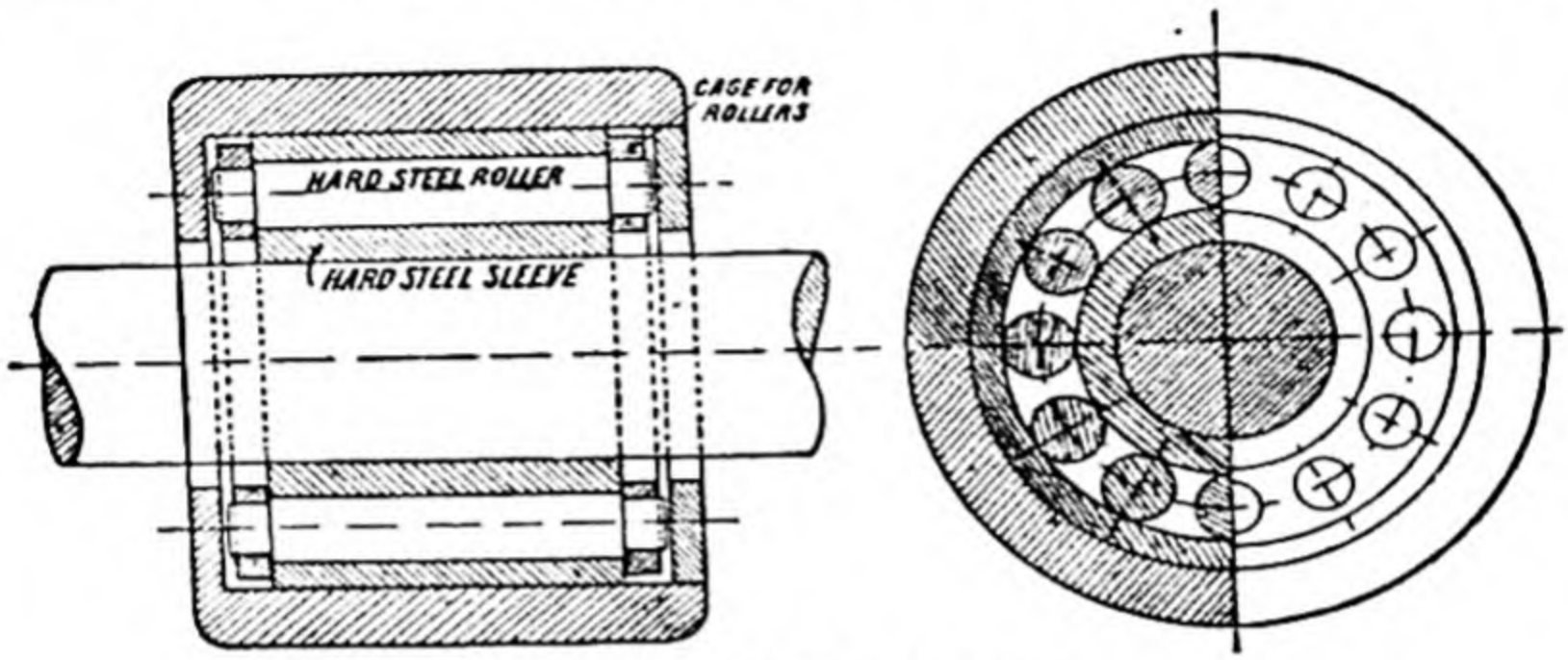

FIG. 45.—Roller bearing.

mechanical friction; for a cart on a good macadam road, the tractive resistance is approximately 30 lb. per ton.

44. Friction and Efficiency of Machines. In many machines, a driving force, which may be called the effort, is exerted, an external source of energy being drawn upon to do useful work in overcoming some resistance or load through some distance not necessarily or generally the same as that through which the effort is exerted. For example, in lifting machines, whether levers, pulley-blocks, screws, spur-geared crabs or other devices, a relatively small effort P may be used to lift a large load W if P moves through a much larger distance than W. The ratio of this distance moved by the effort P to that moved by the load W is the *movement* ratio or, as it is commonly called, the *velocity ratio* V. In many machines, it is constant over finite distances but in others it is variable; and for a given position of the mechanism, the ratio is the limiting value approached when the movements of P and W are indefinitely reduced, or the ratio of the instantaneous velocities of

P and W. It follows from the principle of conservation of energy that if no energy is lost (viz. by frictional resistance converting mechanical energy into heat), the work done against the load W is equal to the work done by P, or for unit movement of W

$$PV = W, \text{ or } P = W/V \quad . \quad . \quad . \quad . \quad (1)$$

But because of friction in actual machines, the load W moved is less than PV; or for a given load W, P must exceed W/V.

The ratio, W/P, is called the *mechanical advantage*. In an ideal frictionless machine, it would be just equal to V, but in actual machines it is less.

The ratio of the work done on the load to that put in by the effort is called the (mechanical) *efficiency* of the machine. For unit displacement of W, the useful work done is W × 1, and the energy expended is PV; hence

$$\text{efficiency} = \frac{W}{PV} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Experimental Results. Experiment shows that for machines with a fixed ratio V between the movement of P and that of W (which may be found by direct measurement or calculated from the data relating to the gearing), the relation of P to W in the form of a graph is a straight line. For example, in a small laboratory machine of which the velocity ratio (V) is 16, the loads and corresponding efforts were as shown in the following table :—

Load W (lb.) .	0	5	10	15	20	25	30	35	40	50
Actual effort P (lb.)	1·5	2·4	3·0	4·2	5·0	5·9	7·0	7·6	9·0	10·5
"Ideal" effort W/16 (lb.) .	0	0·312	0·625	0·937	1·250	1·562	1·875	2·187	2·50	3·125
Efficiency (per cent.) . .	0	13·0	20·8	22·4	25·0	26·5	26·8	28·7	27·8	29·8

The relation of the actual effort P to the load W is shown in Fig. 46, which also shows the ideal effort required for the load in a frictionless machine, *i.e.* W/16. By taking the difference of two values of P for corresponding values of W

it will be apparent that P increases at a rate of 0·18 lb. per lb. increase of W and that it starts from 1·5 lb., namely that of P shown by the graph for W = 0 (not necessarily an actual point determined experimentally with no load, and which may be liable to error). Hence the value of P in lb. is

$$P = 1{\cdot}5 + 0{\cdot}18W \quad . \quad . \quad . \quad . \quad . \quad (3)$$

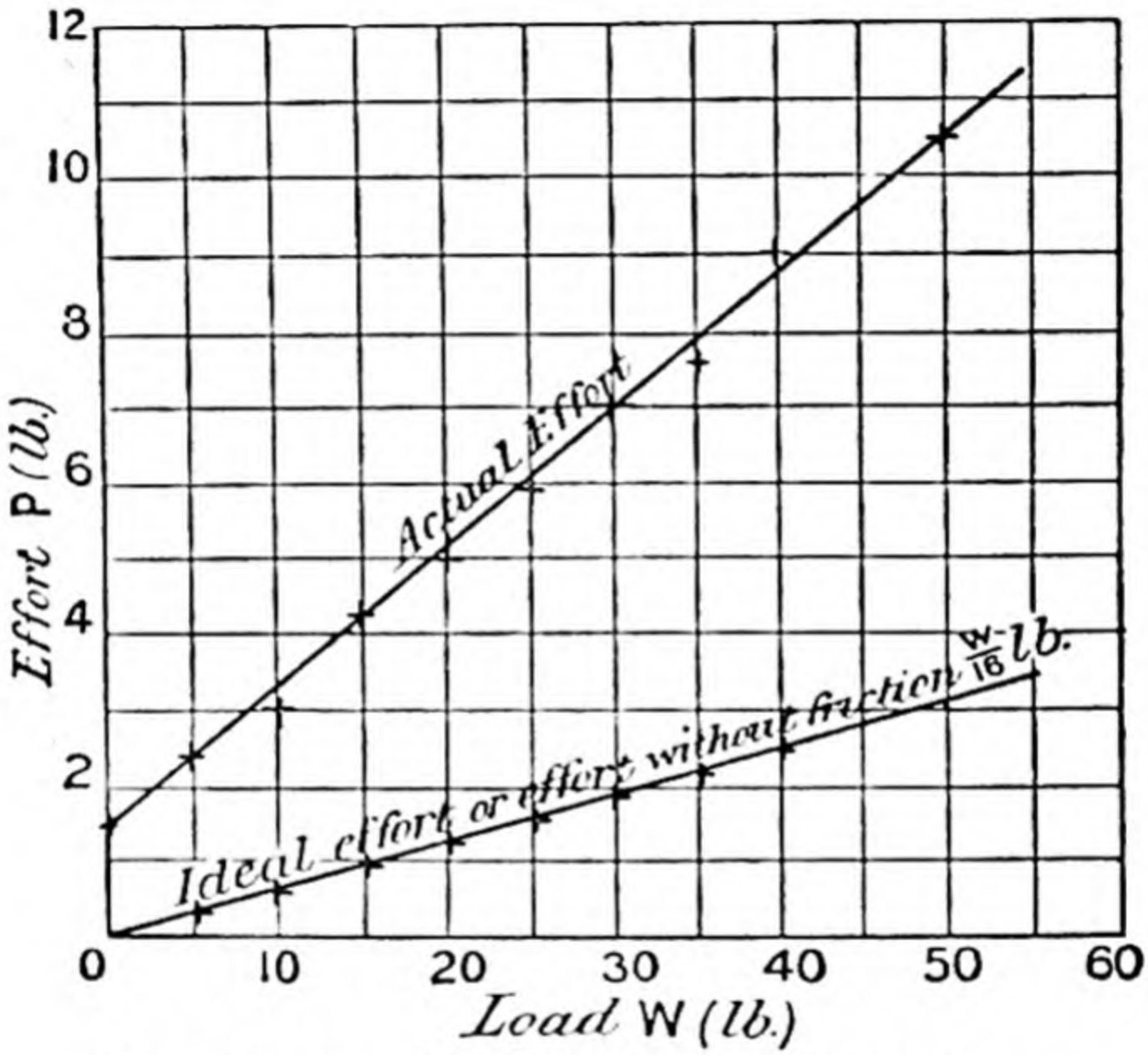

FIG. 46.—Graph of effort for simple machine.

when W is the load in lb. More generally, the effort is

$$P = P_0 + aW \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where P_0 is the value of W from the graph for W = 0, and *a* is the increase of P per unit increase of W.

The graph of the mechanical efficiency (W/PV) is shown in Fig. 47. From equation (3), it is evident that since V = 16

$$\text{Efficiency} = W/PV = W/(1{\cdot}5 + 0{\cdot}18W)16 \quad . \quad (5)$$

If W is very large, this approaches the value

$$W/(16\times 0{\cdot}18W)=1/(16\times 0{\cdot}18)=1/2{\cdot}88$$
$$=0{\cdot}347 \text{ or } 34{\cdot}7 \text{ per cent.}$$

Taking the more general value (4) for P,

$$\text{Efficiency}=W/PV=W/(P_0+aW)V$$

and when W is very large, this approaches the value $1/Va$. The graph is a rectangular hyperbola and approaches a

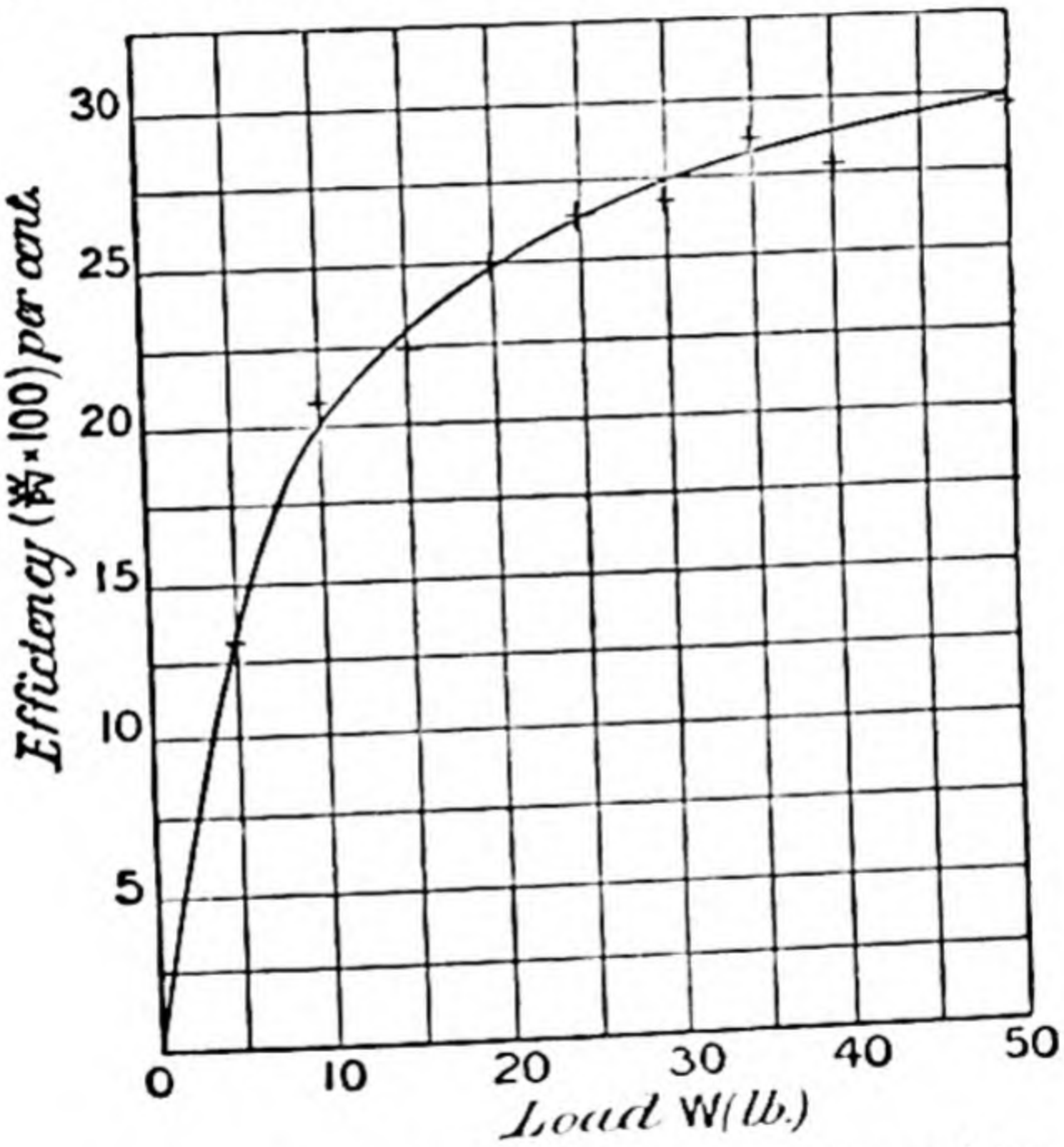

FIG. 47.—Graph of efficiency of simple machine.

horizontal line at a height $1/Va$, which is an asymptote to the curve, *i.e.* a line which would touch the curve as a tangent at an indefinitely great distance. The curve would also approach in the same way a vertical asymptote, for it has a negative infinite value when the denominator is zero, *i.e.* when $aW+P_0=0$ or $W=-P_0/a$, the value of

W for P=0 corresponding to the intersection of the graph of P with the horizontal axis. This negative value of W would imply a reversal in the direction of the force W but not of the direction of motion. And the product of the distances of any point on the graph from the two asymptotes is constant. For the distances are $W+P_0/a$ horizontally and $1/Va-W/(P_0+aW)V$ or $P_0 \div \{Va^2(W+P_0/a)\}$ vertically, and their product is $P_0/(Va^2)$ which is a constant. The equation of a rectangular hyperbola with respect to its asymptotes as axes is that the product of the co-ordinates of any point is a constant. (This is well known in connection with what is called hyperbolic expansion of a gas, *e.g.* expansion according to Boyle's law.) The precise form of this efficiency curve, if the graph of P is a straight line, is of mathematical rather than practical interest.

45. Condition of Reversal of Machine. If a machine is capable of reversal, *i.e.* running back in the opposite direction under the driving force of the load, the efficiency in the normal or forward running direction must be up to a sufficient value. For one foot lift of W, the energy expended is PV and the useful work done in lifting will be $W \times 1$. Hence the energy lost is PV—W. If W is just sufficient to drive the machine with the force (P) against it reduced to zero, W must be just sufficient to overcome the frictional resistance and provide the energy equal to that lost in frictional resistance (and converted into heat energy). Suppose, in reversal, this remains the same, viz. PV—W, as in forward running, though it will actually be a little less with the force W at one end and zero force instead of P at the other. Neglecting this small change,

	W must exceed		PV—W,
	2W	„	PV,
so that	W/PV	„	½ or 50 per cent.

i.e. the efficiency in forward running must exceed 50 per cent. if the machine is able to run backwards or reversed under the influence of W when P is zero. Actually the condition will be that the efficiency W/PV must exceed some value a little less than 50 per cent., allowing for the fact that the frictional work per ft. movement of W will

be a little less than PV—W supposed above. For example, in the case of the inclined plane or screw, where the friction can be stated rather more precisely for running up or down, it was found, (4), Art. 41, that the condition was that the efficiency should exceed $0{\cdot}5(1-\mu^2)$.

In applying the condition for reversal, it should be noted that in the equation (4) of Art. 44, the term P_0, the effort required for steady motion with no load, may include the effort required to lift against gravitation some form of cage or holder which supports the useful load. In reversed running, the weight of such container assists motion instead of being an addition to the load lifted. The foregoing argument applies only to the case in which gravitational effects of parts of the machine or its accessories are eliminated.

46. The Principle of Work and Ideal Machine. In an ideal frictionless machine, running in the equilibrium of constant speed, the work done by the effort is equal to the work done on the load. If motion takes place at constant speed, so that there is no change in the kinetic energy of any moving mass, we may write for an effort P and a load W

$$P \times \text{movement of } P = W \times \text{movement of } W \quad . \quad (1)$$

or
$$\left.\begin{aligned} W/P &= \text{movement of P/movement of W} \\ &= \text{velocity ratio (V)} \end{aligned}\right\} \quad . \quad (2)$$

This principle may be approximately verified experimentally by a machine specially constructed with ball bearings or other means of reducing friction to a very small amount, *e.g.* a knife-edged fulcrum for a simple lever. The principle might also be expressed by stating that for balanced effort P and resistance W there is no accumulation of energy in the machine, *i.e.* reckoning the input and output of energy as of opposite signs, the net work done is zero, and mechanical energy is merely transmitted.

The principle is a special case of the larger principle of the conservation of energy (see Art. 63), which includes the case of accelerated motion with some energy existing in the kinetic form and also takes into account energy which may be converted into heat or other forms of energy.

Taking the wheel and axle, Fig. 48, as an example, and applying the principle of moments (Art. 23) to moments exerted about the axis of the axle, evidently,

$$P \times \tfrac{1}{2}D = W \times \tfrac{1}{2}d \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$PD = Wd \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

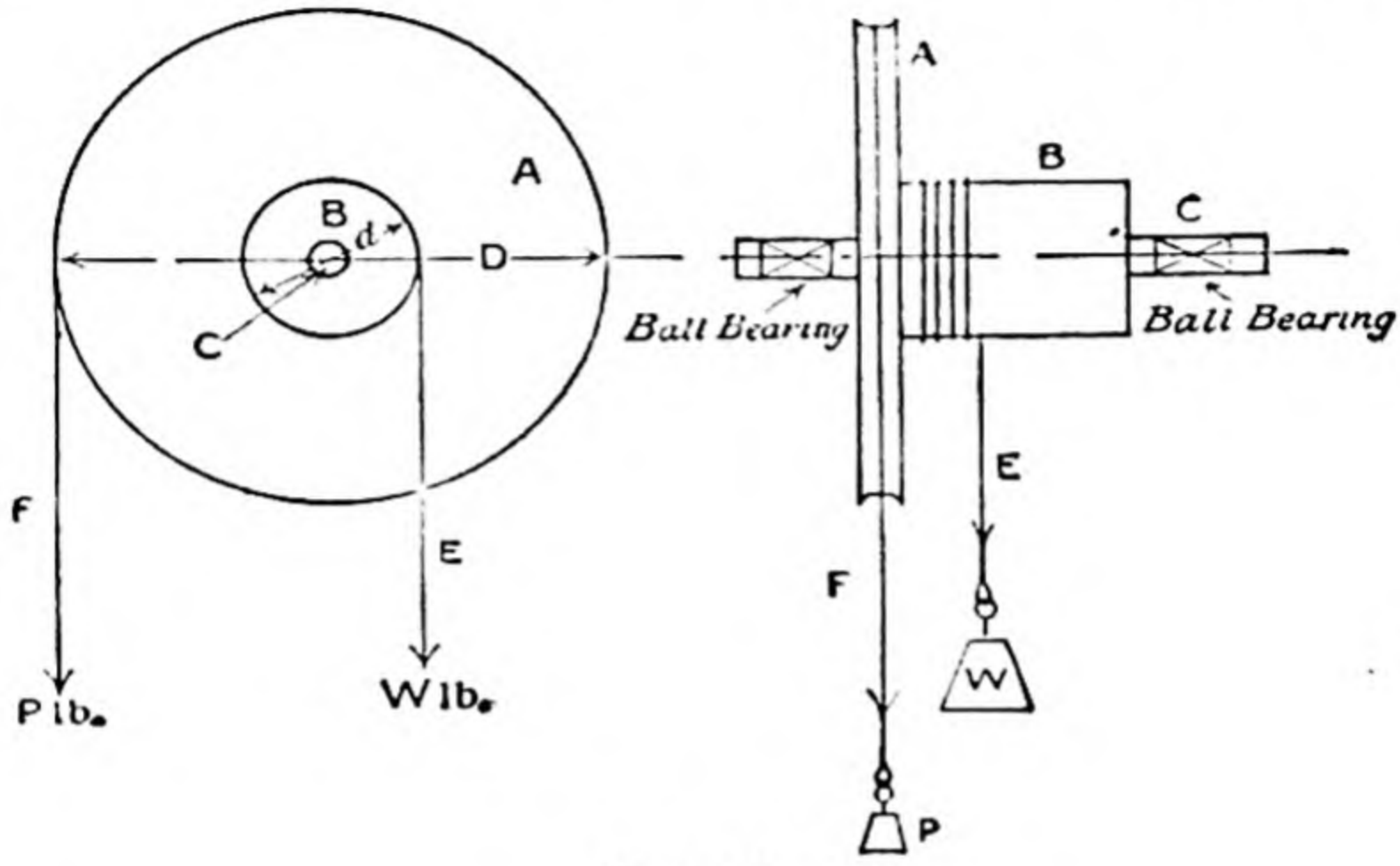

FIG. 48.

And applying the principle of work for rotary motion of the axle through any angle θ

$$P \times \tfrac{1}{2}D \times \theta = W \times \tfrac{1}{2}d \times \theta \quad . \quad . \quad . \quad . \quad (5)$$

which is another statement of equation (1) since $\frac{1}{2}D\theta$ and $\frac{1}{2}d\theta$ are the movements of P and W respectively. (If $\theta = 2\pi$ we have the work done by P and on W per revolution of the axis.) Dividing (5) by $\frac{1}{2}\theta$ gives (4).

Another very simple example of the principle is found in the smooth inclined plane with effort P parallel to the plane. Fig. 49 shows the vector solution in which

$$p/w = P/W = \sin \alpha \quad . \quad . \quad . \quad . \quad (6)$$

while by the principle of work

$$P \times l = W \times h \text{ or } P/W = h/l = \sin \alpha \quad . \quad . \quad (7)$$

Thus the principle of work and the principle of moments (or a vector diagram) lead to the same equations, and either may be directly derived from the other. They are but two ways of stating the same fact. And the problems solved by the usual principles of statics may alternatively be solved by the principle of work, but in many cases there is no advantage in using the principle of work. It has, however, advantages in some cases, when the velocity ratio between the points of application of two forces may be stated or the effect of reactions of some fixed element of a

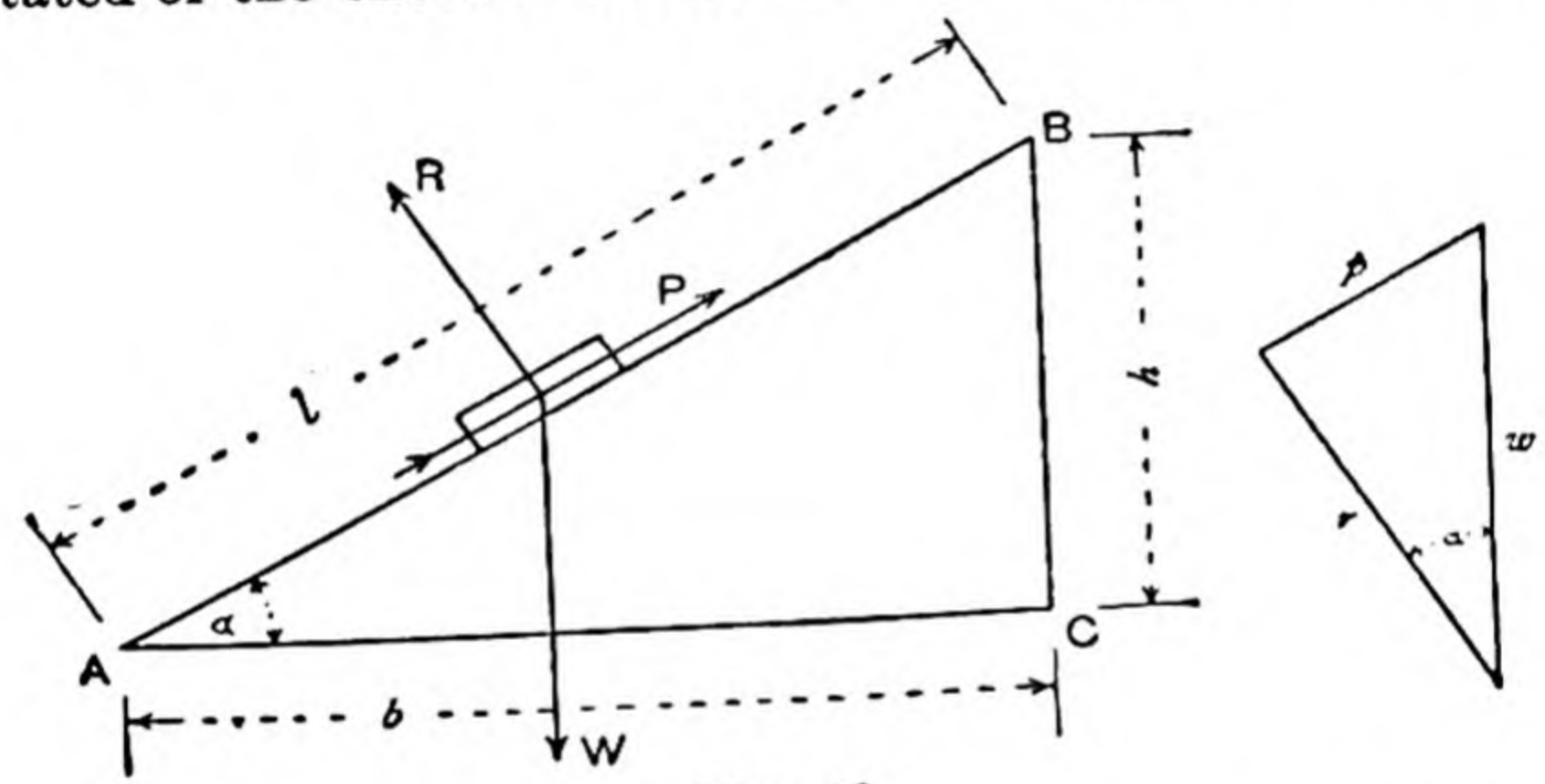

FIG. 49.

part or link may easily be overlooked when using the principle of moments or other statical methods. The wheel and axle chosen as an example, like many machines, has a constant velocity ratio or movement ratio between P and W, but this is not always the case. Some machines have a variable ratio, and in such cases the principle of work, if used, must be applied for indefinitely small movements. As only the *ratio* of movements is required, this presents no difficulty. Frequently the *velocity* ratio is employed. This introduces the element of time, but the ratio of the velocities of two points is the same as the ratio of their movements over an indefinitely small interval of time. But if we state the equation (1) in terms of velocity or speed, it becomes

$$P \times \text{speed of } P = W \times \text{speed of } W \quad . \quad . \quad (8)$$

or in other words

$$\text{power exerted by P} = \text{power exerted on W} \quad . \quad (9)$$

Thus the *rate* of accumulation or reception of energy by the machine is zero, mechanical energy being merely transmitted. It is sometimes stated that the forces exert zero power in the sense that the net input or output of power of all the forces is zero. The introduction of the element of time and statement in terms of *power* instead of *work* is generally unnecessary. The principle of work may be stated in terms of work over finite displacements, or, when the movement or velocity ratio is variable, over indefinitely small displacements.

Since the principle of work and the more usual statical methods of vector force diagrams or moments lead to the same results, the principle of work may be applied to find the velocity ratio between two points of a mechanism at which forces are exerted. For if the ratio of, say, P/W is found by, say, a vector force diagram, then the use of equation (2) will give V, since $P/W = V$; this use of a vector force diagram is illustrated in Example 2 following.

EXAMPLE 1. *Roberval Balance.* The mechanism shown at (*a*) of Fig. 50 consists of bars AB and CD of equal length, pivoted on pins E and F, which are supported in the fixed frame of the balance and connected by two equal vertical links BC and AD, which have horizontal pieces KM and NS carrying weights W_1 and W_2 respectively. Find the relation between the weights W_1 and W_2, the distances a and b and the angle θ for equilibrium.

It is apparent that the vertical links BC and AD remain vertical for any inclination θ and that every point on the link BCKM moves through the same vertical distance when the angle θ is increased from zero to any other value. And if $AE = EB = l$, all points in the link ADNS move the same vertical distance as those in BCKM. Hence the velocity ratio of the weight W_2 to W_1 is unity, and $W_2 = W_1$ whatever the values of a, b, or θ. If W_1 is moved along KM or W_2 is moved along SN, the equilibrium is unaffected.

If the arms of the balance are unequal, say, $AE = DF = nl$ while $EB = FC = l$, then W_2 moves n times as far as

W_1. If W_1 has a vertical movement $l \sin \theta$, then W_2 has a vertical movement $nl \sin \theta$, and by the principle of work

$$W_2 nl \sin \theta = W_1 l \sin \theta$$

or
$$W_1/W_2 = n \text{ or } W_1 = nW_2$$

which is again independent of the lengths a and b and the angle θ.

The principle of work thus affords a very simple solution of this problem. The solution by the usual conditions of equilibrium (Art. 26) is not quite so simple. But it provides an instructive example of what can be found by applying the conditions in turn to the several component

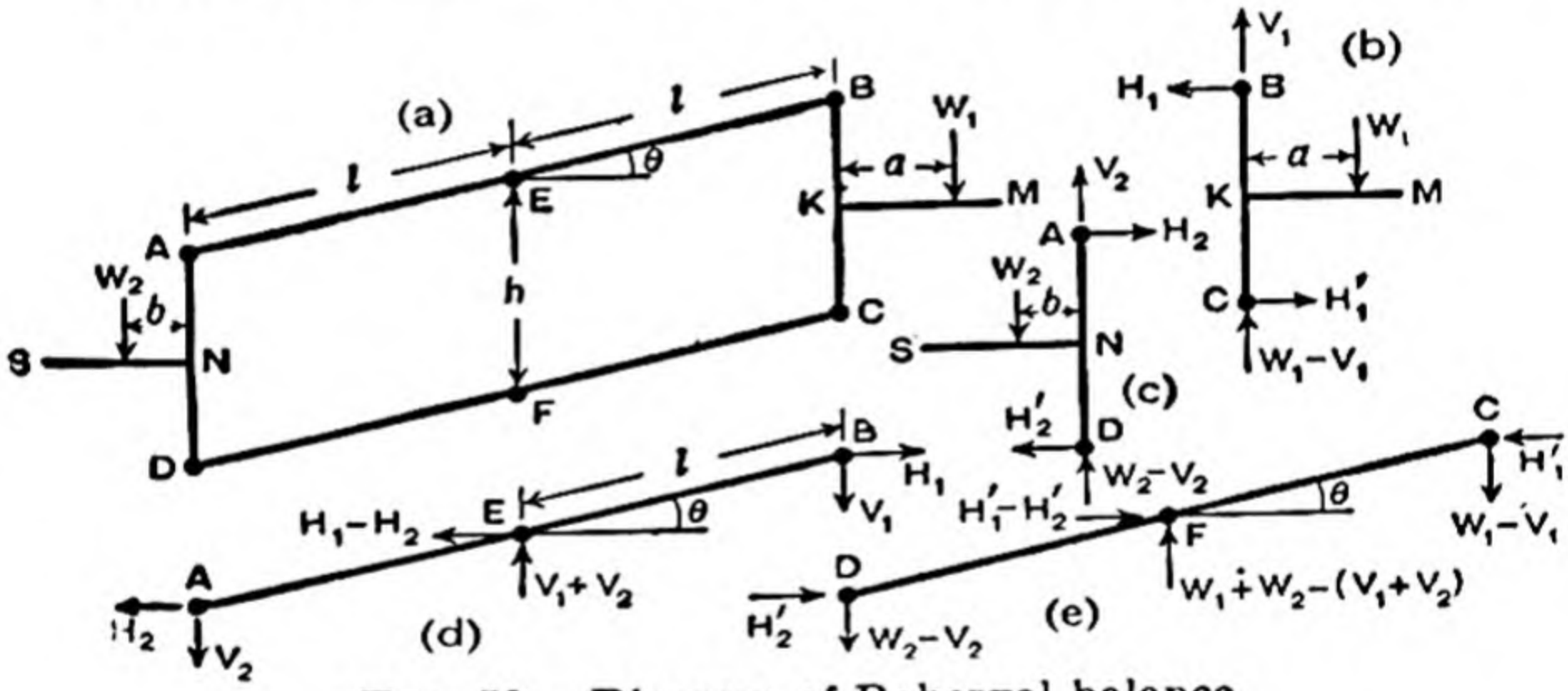

FIG. 50.—Diagram of Roberval balance.

rigid pieces into which a mechanism can be divided. The four parts of the mechanism are shown separately in Fig. 50, with the forces on them resolved into horizontal and vertical components.

Consider the link BCKM as shown detached at (*b*) of Fig. 50 with the components of the forces exerted upon it by the pins at B and C. If the vertical upward component at B is V_1, that at C must be $W_1 - V_1$, to balance W_1 acting downward; and H_1' at C must be equal to H_1 at B (but in opposite direction) since these are the only horizontal forces on the link. Thus

$$H_1' = H_1 \text{ and similarly } H_2' = H_2 \quad . \quad . \quad (10)$$

The forces exerted on AB and DC at the pins are the equal and opposite thrusts (reactions) to those exerted on the vertical links at the pins. (The pins may be regarded as part of either the vertical links or the horizontal bars, and free to turn in holes in the other member.) Hence the forces exerted on AB at the pins A and B are as shown at (*d*), Fig. 50. The (reactive) forces exerted by the fixed pins E and F are shown at (*d*) and (*e*); but it is not necessary to know their magnitudes, as they have no moment about the pins, about which equations of moments can conveniently be formed. Thus considering the equilibrium of the bars AB and DC in turn, from (clockwise) moments about E of forces exerted on AB,

$$(H_1+H_2)l \sin\theta+(V_1-V_2)l\cos\theta=0$$

or
$$V_1-V_2=-(H_1+H_2)\tan\theta \quad . \quad . \quad . \quad (11)$$

and from clockwise moments about F of forces on DC,

$$-(H_1+H_2)l\sin\theta+\{W_1-V_1-(W_2-V_2)\}l\cos\theta=0$$

or
$$-V_1+V_2+W_1-W_2=(H_1+H_2)\tan\theta \quad . \quad (12)$$

Adding (11) and (12)

$$W_1-W_2=0 \text{ or } W_1=W_2$$

whatever the values of a, b, and θ, which is the result previously reached by the principle of work. And again, if $AE=DF=nl$ instead of l, the equations (11) and (12) are slightly modified and

$$W_2nl=W_1l, \text{ or } W_1/W_2=n$$

as before.

In many cases a solution of a problem from the usual conditions of equilibrium (Art. 26) gives more complete information about the forces acting in the mechanism, but in this example the vertical components V_1 and V_2 are indeterminate although from moments about, say, K and N, it is evident that $H_1=W_1a/h$ and $H_2=W_2b/h$. If the point E is slightly raised, the bar AB may carry the whole vertical load $V_1=W_1$ and $V_2=W_2$. Or if E is slightly lowered, V_1 and V_2 may each become zero.

Instead of the two equations (11) and (12) expressing the facts that the moments on the two arms separately

are zero, we might have applied the condition of zero moments for all the (external) forces exerted on the whole mechanism about either of the pivots E or F. About E, say, the forces are W_1, W_2, and the force exerted by the pivot F, the vertical component of which has zero moment about E. Thus the (clockwise) moments about E are

$$W_1(a+l\cos\theta)-W_2(b+l\cos\theta)-(H_1-H_2)h=0 \quad . \quad (13)$$

and substituting for H_1h and H_2h their values W_1a and W_2b respectively, this gives

$$(W_1-W_2)l\cos\theta=0 \text{ or } W_1=W_2 \quad . \quad . \quad (14)$$

irrespective of a, b, and θ as before.

In treating this problem by the condition of zero moment about one pivot, of all the external forces on the mechanism, the fact that the other pivot exerts a reaction or supporting force may be overlooked, while by solving the problem by the principle of work this risk does not occur. Nevertheless the solution by the normal conditions of equilibrium is instructive. It teaches us that reactions at supports should *not* be overlooked.

EXAMPLE 2. *Toggle Joint.* In this arrangement of links shown in Fig. 51, a moderate force or effort P is exerted against a much larger load W. Find the relation of W to P and hence find the velocity ratio of P to W.

P acts horizontally at the pin X connecting the two inclined links ZX and XY. The pin Y at the end of the link XY is in a block which slides in guides of negligible friction so that the reaction or force exerted on the block by the guide is perpendicular to the guide. W is the vertical (upward) force exerted on the sliding block by the resistance, an equal (downward) thrust W being exerted on the load or resistance by the block. Let ZX=XY $=l$, say. Draw the simple vector force diagrams *abc* and *cbd* for the balanced forces acting on the pins X and Y respectively. The forces are shown in space and vector diagrams by Bows' notation (Art. 31). If the angle $\mathrm{Y\hat{Z}X}$ $=\mathrm{X\hat{Y}Z}=\theta$, evidently $b\hat{c}a=2\theta$ and $c\hat{b}d=\theta$,

$$P=ab=2bc\sin\theta, \text{ or } bc=\tfrac{1}{2}P/\sin\theta$$

and

$$W = bd = bc \cos\theta = \tfrac{1}{2}P \cos\theta / \sin\theta = \tfrac{1}{2}P \cot\theta$$

or

$$W/P = \tfrac{1}{2}\cot\theta.$$

Thus as θ becomes very small, W/P becomes very large, approaching an infinitely large value as θ approaches zero, but only on the assumption of rigid or inelastic material (which does not exist).

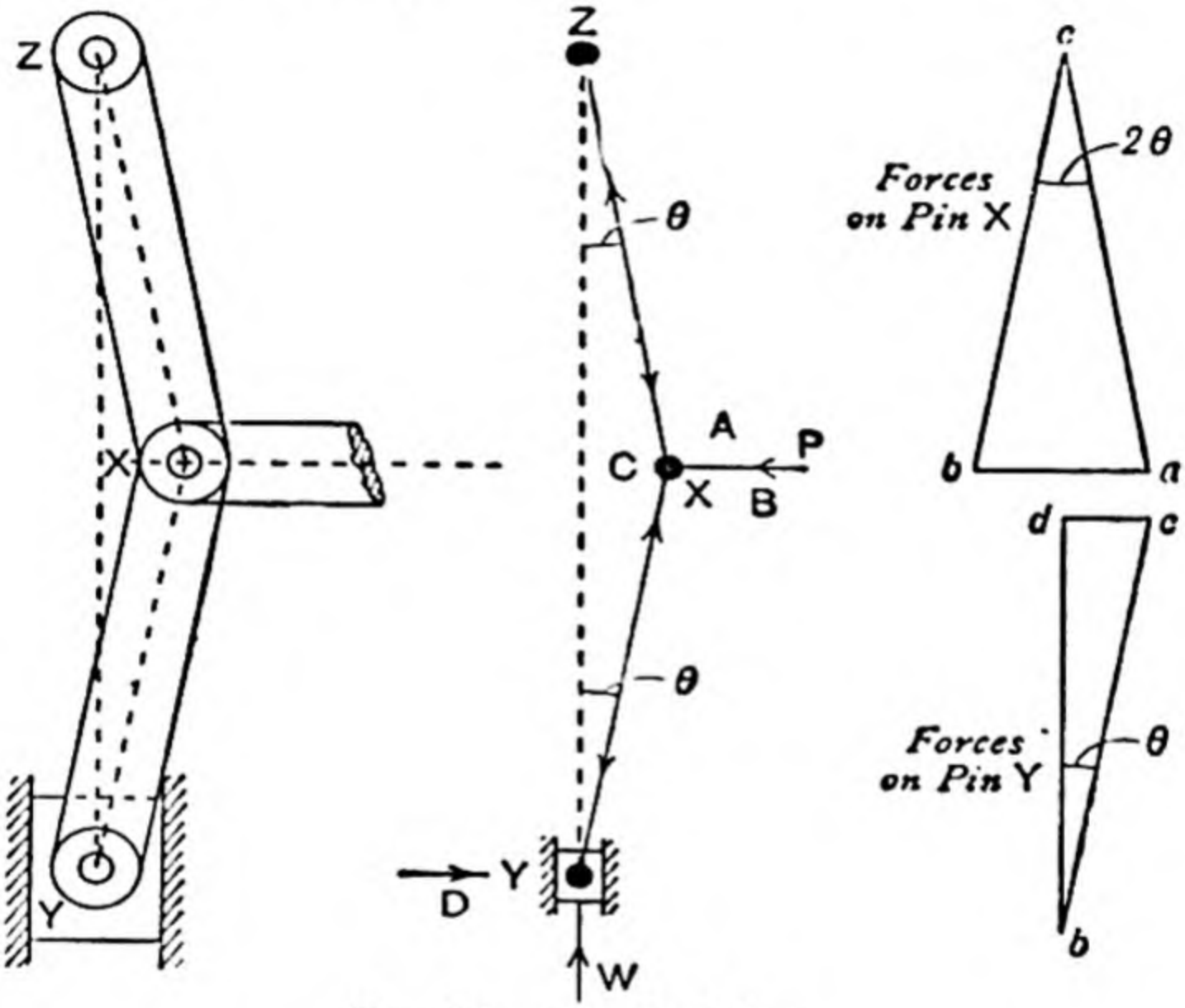

FIG. 51.—Toggle joint.

From the principle of work,

P × horizontal velocity of P = W × velocity of W,

hence

Velocity ratio = velocity of P/velocity of W = W/P = $\tfrac{1}{2}\cot\theta$

the large ratio W/P corresponding to the large motion of P compared to that of W. It is convenient to speak of the velocity of P, but more accurately we should say the

horizontal component velocity of the point of application X of the (horizontal) force P.

We may note a type of solution of the force ratio P/W by the principle of work in which the velocity ratio is found from small changes of position of the pins X and Y, which give the movement ratio. Let the distance of X from the fixed line ZY be x and the distance of Y from the fixed point Z be y. Then for a small *increase* $\delta\theta$ in θ, x increases by an amount δx say, which is $\delta\theta \times dx/d\theta$; and since $x = l \sin\theta$, $dx/d\theta = l\cos\theta$. And for an increase $\delta\theta$, the increase in y is $\delta\theta \times dy/d\theta$, and since $y = 2l\cos\theta$, $dy/d\theta = -2l\sin\theta$ (a decrease in y with increase in θ), and the ratio of these two small changes is

$$\text{movement of P/movement of W} = \delta x/\delta y$$
$$= l\cos\theta \,.\, \delta\theta/2l\sin\theta \,.\, \delta\theta$$
$$= \tfrac{1}{2}\cot\theta.$$

Or to state the equation of work for the small displacements

$$P \times \delta\theta \times l\cos\theta = W \times \delta\theta \times 2l\sin\theta$$
$$W/P = \tfrac{1}{2}\cot\theta.$$

It should be noted that the velocity ratio applicable to the principle of work is that of the displacement or speed of the point of application of the force *resolved in the direction of the force*, and the displacements in the equation of work are those in the directions of the forces. Thus in Fig. 51, the actual direction of movement of the pin X is not horizontal but δx is the horizontal component of that movement. (The displacements δx and δy may also be found from the geometry of the figure by noting the displacements of X and Y for a small change $\delta\theta$ in θ instead of by differentiating the expressions for x and y.)

We shall note another way of finding the velocity ratio of X to Y in Art. 54.

EXAMPLE 3. *Rapson's Slide.* This mechanism for putting over the tiller and turning the rudder of a ship is shown diagrammatically in Fig. 52. A block a works in a fixed slide and carries a pin B about which a second block b can turn while it can also slide on the tiller t which turns

about a fixed axis A, thereby turning the rudder. When a force P is applied to the block *a*, what torque about A is exerted on the tiller, and for a given linear speed of the block *a*, what is the angular velocity of the tiller?

Let P be the force exerted on the block *a* in the direction CB and let Q be the resisting force exerted by the tiller on the block *a* through the pin B and block *b*. This must be perpendicular to the tiller *t* if friction is negligible.

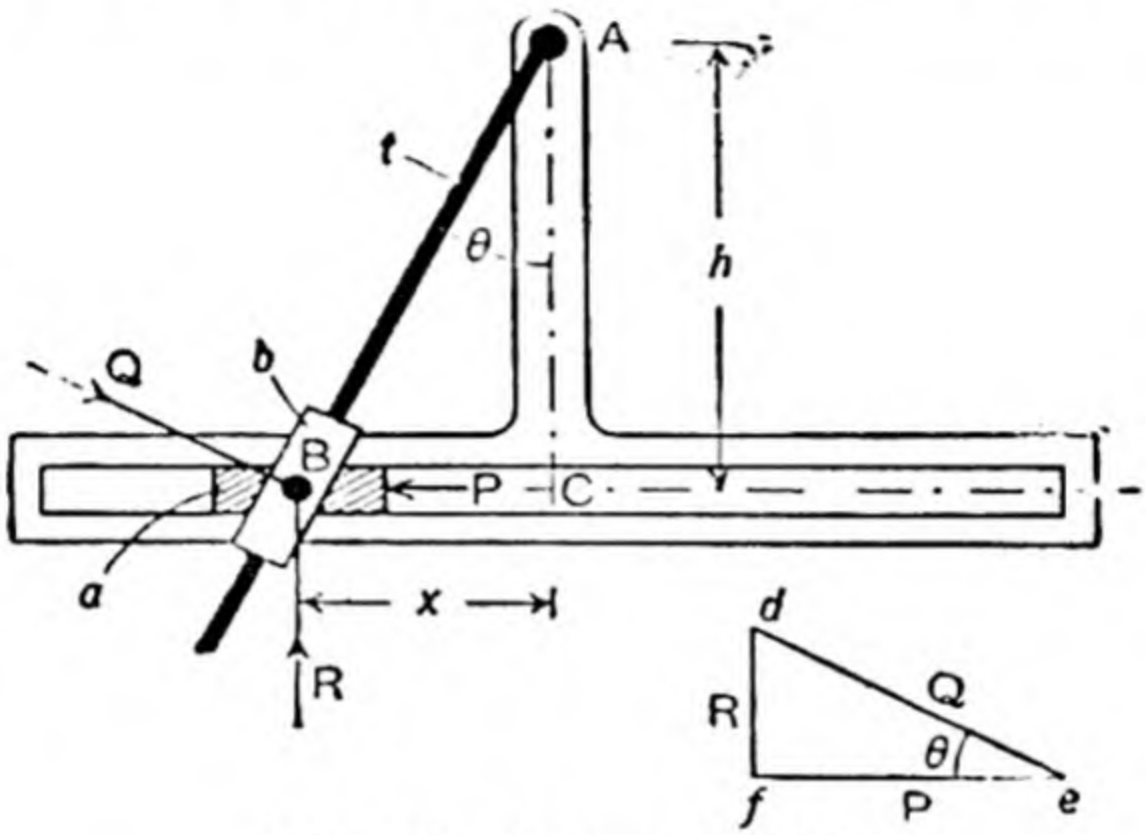

FIG. 52.—Rapson's slide.

Evidently P and Q, not in line, cannot balance one another and a third (supporting) force must act on the block *a*. And if friction is negligible, this force R will be perpendicular to the slide, *i.e.* perpendicular to BC as shown. The three concurrent forces P, Q, and R exerted on the slide *a* are in equilibrium and they are shown in the triangular vector force diagram *def*, from which it is evident that $Q = P \sec \theta$. And if $AC = h$, evidently $AB = h \sec \theta$. Hence the torque on the tiller is

$$Q \times AB = P \sec \theta \times h \sec \theta = Ph \sec^2 \theta$$

a value which increases with θ from the value Ph when $\theta = 0$, and the torque increases as the helm is put over.

And from the principle of work,

$$\text{Velocity of Q/velocity of P} = P/Q = \cos \theta$$
$$\text{Velocity of Q (and B)} = v \cos \theta$$

where v is the velocity of the block a.

If ω is the angular velocity of the tiller about A,

ω = velocity of B/AB = $v \cos\theta/h \sec\theta = v(\cos^2\theta)/h$.

In solving this problem by the usual principles of equilibrium, there is a liability to ignore the existence of the reaction R of the slide on the block, while in solving by the principle of work the reaction R does not appear. Thus if $BC = x = h \tan\theta$, $dx/d\theta = h \sec^2\theta$, and for small displacements δx and $\delta\theta$, if T is the torque on the tiller, by the principle of work

$$P \,.\, \delta x = T \,.\, \delta\theta$$

$P(dx/d\theta)\delta\theta = T\delta\theta$, and substituting for $dx/d\theta$

$P \,.\, h \sec^2\theta \,.\, \delta\theta = T \,.\, \delta\theta$ or $T = Ph \sec^2\theta$ as before.

Also $v = dx/dt$, and $\omega = d\theta/dt = (dx/dt) \div (dx/d\theta) = v/h \sec^2\theta$
$= v(\cos^2\theta)/h$ as before.

EXAMPLES III

1. The crank of an engine is 9 inches long, and the connecting-rod 3 feet long. When the crank is at right angle to the line of stroke, the total force on the piston is 8000 lb. If the coefficient of friction between the crosshead slipper and guides is 0·05, find the frictional force opposing the sliding of the crosshead in this position.

2. The slope of an inclined plane is 20° to the horizontal. A body weighing 2 cwt. rests on the plane. What force inclined 15° to the plane will be necessary to draw the body up the plane?

3. A tramcar weighs 6 tons, the tractive resistance on the level being 20 lb. per ton. What horse-power will be required to propel the car at a uniform speed of 12 miles an hour, (*a*) up an incline of 1 in 15, (*b*) on the level. Take the efficiency of the motors and drive as 70 per cent.

4. A screw-jack has a thread of $\frac{3}{8}$-inch pitch. What effort applied at the end of a handle 15 inches long will be required to lift a load of 2 tons, the efficiency at this load being 45 per cent.?

5. A screw-jack has a double thread, the thickness of the square thread being $\frac{1}{4}$ inch. If the lever is 20 inches long, what is the velocity ratio?

6. The table of a planing machine is driven by a screw of 2 inches pitch. The driving pulley on the screw is 20 inches diameter and the difference of tensions of the belt on it is 175 lb. The coefficient of friction between the table and its guides is 0·10 and

the efficiency of the screw is 40 per cent. : find the total weight of the table and the work on it.

7. The efficiency of a screw is 55 per cent. and its pitch 3 inches. What will be the axial thrust of the screw when a twisting moment of 4 ton-inches is applied to it ?

8. In a right- and left-handed screw coupling for railway carriages the pitch of the two screws is $\frac{1}{8}$ inch. The lever attached to the nut is 18 inches long and a force of 40 lb. is exerted on the end of it. With what force will the two carriages be drawn together ?

9. What must be the speed in revolutions per minute of a single-threaded worm driving a worm wheel of 45 teeth at a speed of 150 revolutions per minute ?

10. The following results were obtained from an experiment with a certain lifting machine whose velocity ratio is 24·9 :—

Load W lb. .	0	5	10	15	20	25	30	35	40
Effort P lb. .	0·094	0·45	0·81	1·17	1·53	1·88	2·25	2·61	2·97

Find the efficiency of the machine when lifting loads of (*a*) 8 lb., (*b*) 18·5 lb., (*c*) 26 lb.

11. The results shown in the following table were obtained experimentally. Plot the curve of P and W and find its equation.

Load W (lb.) .	0	5	10	15	20	25	30	35	40
Effort P (lb.) .	0·094	0·45	0·81	1·17	1·53	1·88	2·25	2·61	2·97

12. A bicycle is so geared that for one revolution of the crankshaft the cycle travels a distance equal to the circumference of a wheel of 70 inches diameter. The cranks are 6·5 inches long from the axis of the crankshaft to the axis of the pedal spindle. If a rider weighing 140 lb. exerts a vertical force equal to his weight on one pedal when the crank is horizontal, find the tangential thrust between the road and the tyre of the driving wheel. If the cycle weighs 29 lb., find the instantaneous acceleration of the machine and rider on level ground, neglecting any resistances and the rotatory inertia of wheels and cranks.

13. If, in a toggle joint such as is shown in Fig. 51, the effort P is replaced by a torque T applied to the link ZX turning about the fixed pin Z with angular velocity ω, find the speed of the pin Y for any angle θ if $XZ = XY = l$, and find the magnitude of W for any position (neglecting the inertia of all moving parts).

CHAPTER IV

KINEMATICS

47. Motion of a Point. Kinematics deals with motion—displacement, velocity, and acceleration—without reference to the forces causing motion. It is assumed that the reader has some elementary knowledge of the motion [1] of a point summarized in Arts. 2 and 3. We now proceed to amplify this statement.

48. Rectilinear Motion. When the motion is of a simple kind, such as at constant speed or with uniform acceleration, ordinary arithmetical calculation or numerical substitution in an algebraic formula (Art. 3 or Art. 15) is usually the quickest mode of finding, say, the distance moved or the acceleration from given speed at stated intervals of time. But when a point or a body has a velocity varying in some way not recognizable as following a simple algebraic expression, graphical calculation (or arithmetical tabulation which can well be illustrated graphically) may be employed. Such computations will be best understood from actual numerical examples as follows :—

EXAMPLE 1. A car starting from rest has velocities v feet per second after t seconds from starting, as given in the following table :—

t	0	4	9	17	24	30	32	40	45	53	58	62
v	0	11·0	22·6	35·6	44·5	49·0	48·9	40·6	33·7	26·8	24·3	24·0

Find the accelerations at all times during the first 60 seconds, and draw a curve showing the accelerations during this time. Find also the distance covered by the car at all times during the first 60 seconds and the average speed in that time

[1] Such as given in *Mechanical Engineering Science* (Morley and Hughes), Chapter I, or in *Elementary Applied Mechanics* (Morley and Inchley).

First plot the curve of velocities on squared paper from the given data, choosing suitable scales. This has been done in Fig. 53, curve I, the scales being, say, 1 inch to 10 seconds and 1 inch to 20 feet per second.

In the first 10 seconds, RQ represents 24·2 feet per second gain of velocity, and OQ represents 10 seconds ; therefore

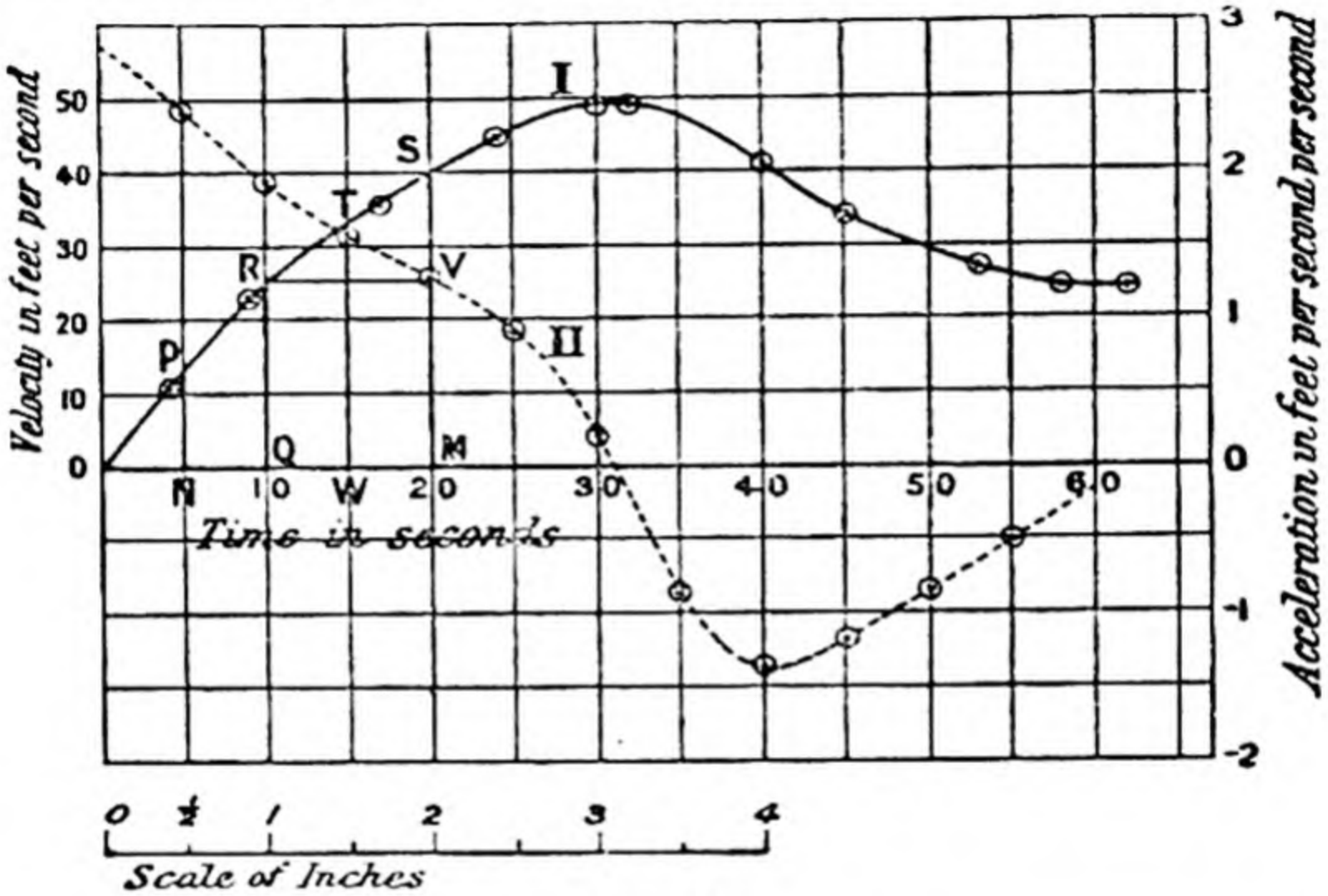

FIG. 53.

the acceleration at N, 5 seconds from starting, is approximately 24·2/10, or 2·42 feet per second per second, corresponding to the gradient at P. Or thus : unit gradient 1 inch vertical in 1 inch horizontal represents

$$\frac{20 \text{ feet per second}}{10 \text{ seconds}} = 2 \text{ feet per second per second}$$

From Fig. 53 $$\frac{RQ}{OQ} = \frac{1{\cdot}21 \text{ inch}}{1 \text{ inch}} = 1{\cdot}21$$

hence acceleration at N $= 1{\cdot}21 \times 2$

$= 2{\cdot}42$ feet per second per second.

Similarly, in the second 10 seconds, SV, which is SM − RQ, represents (39·8 − 24·2), or 15·6 feet per second gain of

velocity; therefore the acceleration at W, 15 seconds from starting, is approximately 15·6/10, or 1·56 feet per second per second. This corresponds very closely to the gradient at T.

Continue in this way, finding the acceleration at say 5, 15, 25, 35, 45, and 55 seconds from starting; and if greater accuracy is desired, at 10, 20, 30, 40, 50, and 60 seconds also. The simplest way is to read off from the curve I velocities in tabular form, and by subtraction find the increase, say, in 10 seconds, thus

t :	0	5	10	15	20	25	30	35	40	45	50	55	60
v :	0	13·5	24·2	32·8	39·8	45·4	49·0	47·5	40·6	33·7	29·0	25·1	24·1
Change in v for 10 secs.		24·2	19·3	15·6	12·6	9·2	2·1	−8·4	−13·8	−11·6	−8·6	−5·1	
Acceleration		2·42	1·93	1·56	1·26	0·92	0·21	−0·84	−1·38	−1·16	−0·86	−0·51	

From the last line in this table, curve II, Fig. 53, has been plotted, and the acceleration at any instant can be read off from it.

In the first 10 seconds the distance covered is found approximately by multiplying the velocity after 5 seconds by the time, *i.e.* 13·5 × 10 = 135 feet. This approximation is equivalent to taking 13·5 feet per second as the mean velocity in the first 10 seconds.

In the next 10 seconds, the mean velocity being approximately TW, which scales 32·8 feet per second (corresponding to t = 15 seconds), the distance covered is 32·8 × 10 = 328 feet, therefore the total distance covered in the first 20 seconds is 135 + 328 = 463 feet. Proceeding in this way, taking 10-second intervals throughout the 60 seconds, and using the previous tabulated results, we get the following table:—

t . . .	0	10	20	30	40	50	60
Distance in previous 10 secs.	0	135	328	454	475	337	251
Total distance.	0	135	463	917	1392	1729	1980

The curve of displacements, Fig. 54, has been plotted from these values.

Greater accuracy may be obtained by finding the distance moved every 5 instead of every 10 seconds.

$$\text{The average velocity} = \frac{\text{displacement}}{\text{time taken}} = \frac{1980}{60}$$

$$= 33{\cdot}0 \text{ feet per sec.}$$

Note that this would be represented on Fig. 53 by a height which is equal to the total area under curve I divided by the length of base corresponding to 60 seconds.

In the foregoing solutions the graphs are illustrations to

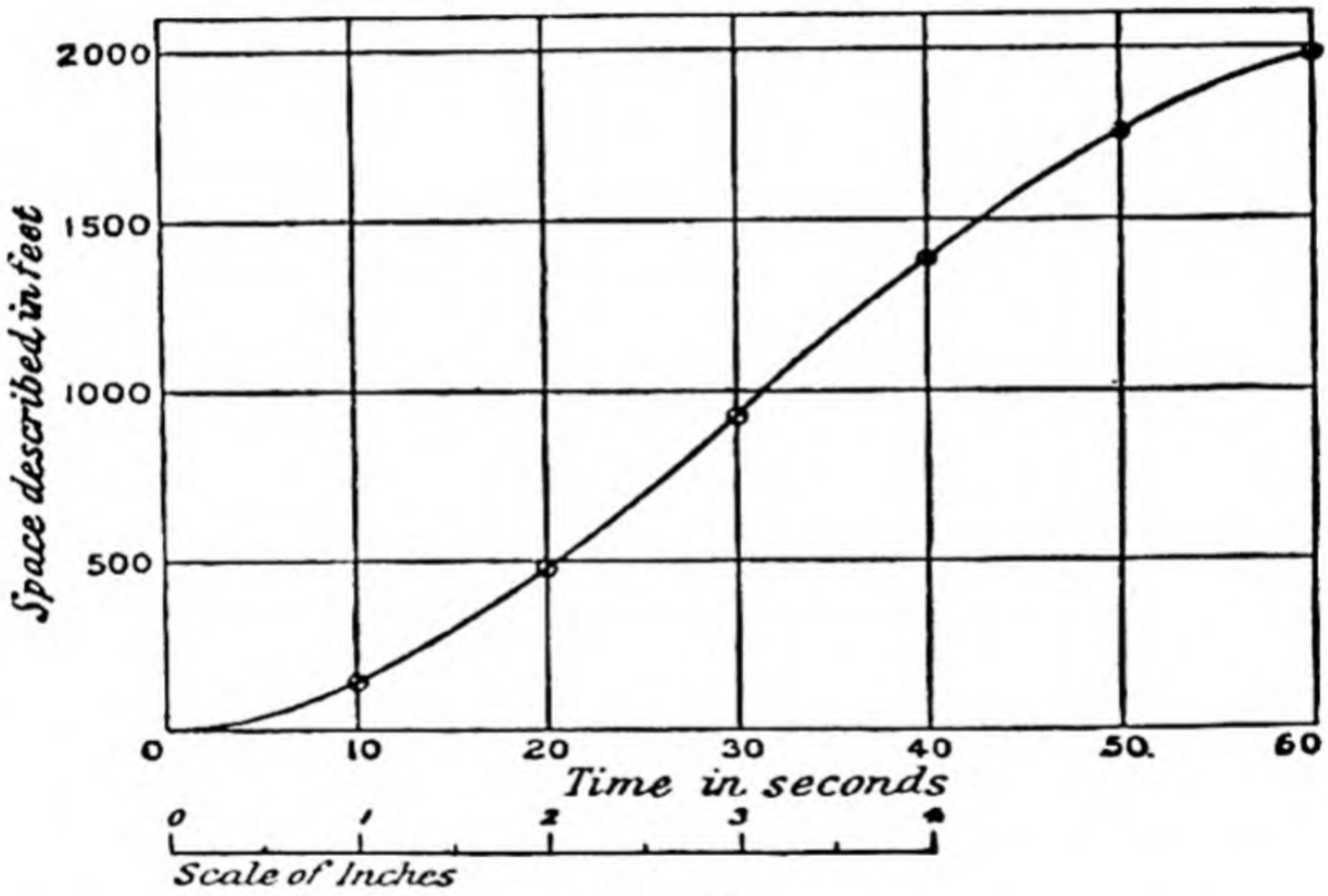

FIG. 54.

explain arithmetical tabular work. A more definitely graphical solution would be to measure by any means the area under curve I (Fig. 53) from the origin to any vertical ordinate such as SM and find what distance this represents by observing what distance is represented by one square inch of area. In this case (Fig. 53) since 1 inch horizontally represents 10 seconds, and 1 inch vertically represents 20 feet per second, one square inch represents 10 sec. $\times$ 20 ft./sec. = 200 ft.

In finding the distance covered we have in effect taken the areas of vertical strips under the curve to be equal to their widths multiplied by the vertical heights of their middle ordinates, *e.g.* for $t=10$ to $t=20$ sec., we took QM $\times$ TW or 10 sec. $\times 32{\cdot}8$ ft. per sec. $=328$ ft. When the curve is convex upwards this result will be slightly too large, and when concave upwards it will be slightly too small. If we had taken heights of the strips to be the arithmetic means of the heights of the vertical boundaries of the strips, the reverse would have been true, *e.g.* for $t=10$ to $t=20$ sec., the arithmetic mean $=\frac{1}{2}(24{\cdot}2+39{\cdot}8)=32{\cdot}0$ ft. per sec., and the distance covered would by such computation be 10 sec. $\times 32$ ft./sec. $=320$ ft., which for a curve convex upward is too low.

EXAMPLE 2. A point moves from rest with the accelerations f in ft. per sec. per sec. given in the following table for the times t seconds from starting.

t	0	4	8	12	16	20
f	10	9·6	8·4	6·4	3·6	0

Find the speed of the point at the end of each 4-second interval and the distances traversed.

The acceleration-time graph is plotted from the table in Fig. 55. Reading accelerations f from the curve, the values at the middles of the five vertical strips corresponding to 4-second intervals are as follow in the table and represent mean accelerations for such periods.

t	2	6	10	14	18
f	9·9	9·1	7·5	5·1	1·9

And the added speed in each 4-second interval is found by multiplying the mean acceleration by 4 seconds, *e.g.* in the second interval ($t=4$ to $t=8$ sec.) the added speed is 9·1 ft./sec.^{2} $\times 4$ sec. $=36{\cdot}4$ ft. per sec. This, added to the

speed of 39·6 ft./sec., gives 76·0 ft./sec. at the end of 8 seconds.

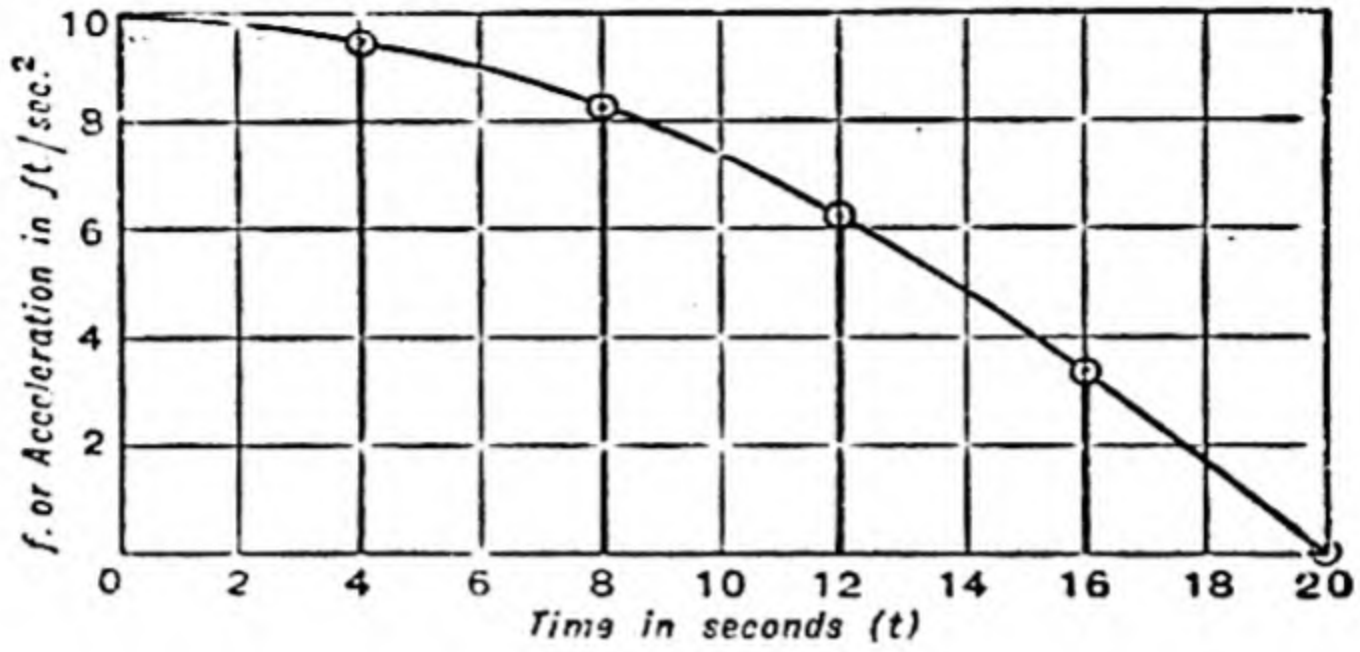

FIG. 55.

t . . .	4	8	12	16	20
Increase of v in previous 4 secs.	39·6	36·4	30·0	20·4	7·6
v . . .	39·6	76·0	106·0	126·4	134·0

The variation of v is shown on the speed-time graph plotted from this table in Fig. 56.

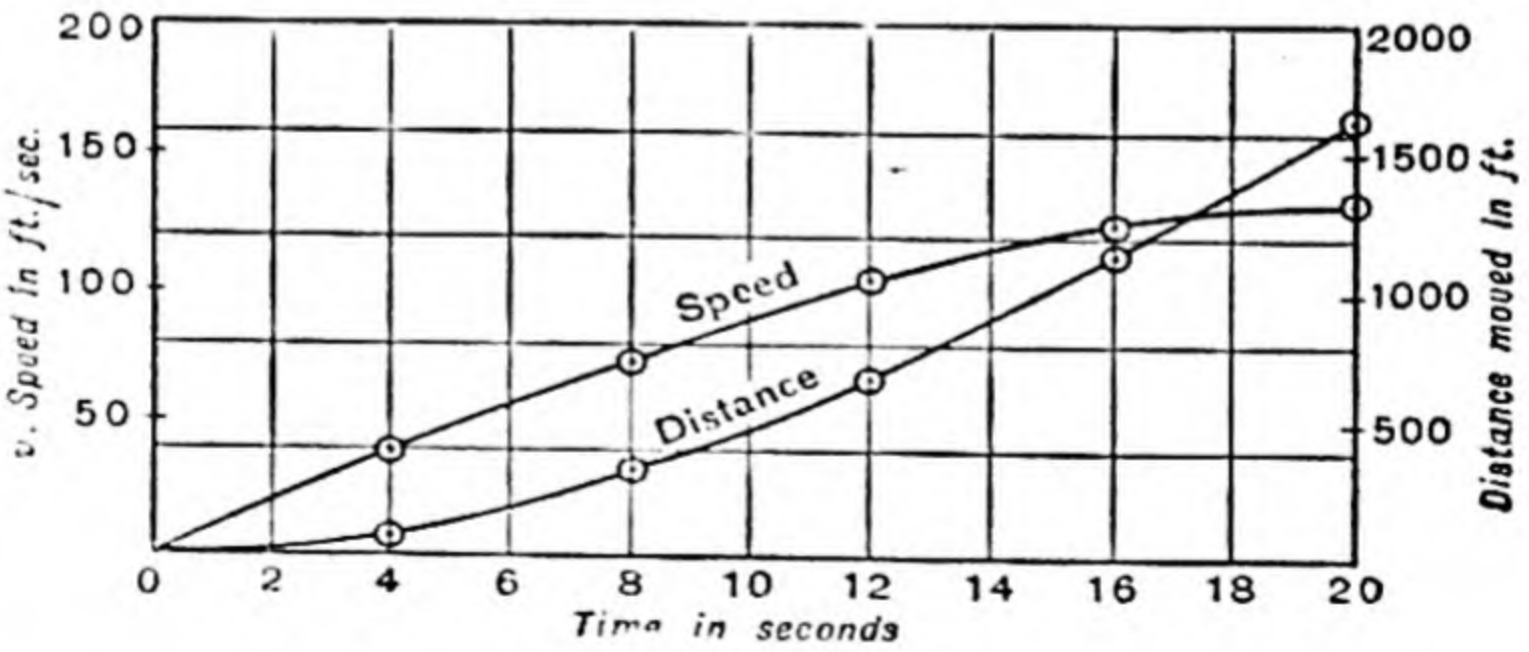

FIG. 56.

By plotting the graph of v to a reasonably large scale, we can scale off the speeds at the middle of each 4-second

interval, and taking these as the mean speed find the distance traversed in each 4-second interval. But if we take the mean speeds as the arithmetic means of the speeds at the beginnings and ends of the intervals we should find the following values of the mean speed and distances covered.

t . . .	4	8	12	16	20
Mean speed in ft./sec. .	19·8	57·8	91·0	116·2	130·2
Distance moved in 4 sec. .	79·2	231·2	364·0	464·8	520·8
Total distance .	79·2	310·4	674·4	1139·2	1660·0

The distances moved are shown in the graph in Fig. 56. Note that at the start, when the acceleration is changing only slowly, the speed-time graph is nearly straight; that its slope falls to zero, with the acceleration, at $t=20$ seconds; that the distance or displacement-time graph approaches straightness at a constant slope as the speed-time curve approaches a constant value at $t=20$ sec.

In estimating the values of v from the middle ordinate of a curve of upward convexity we have values slightly too high. Had we taken instead ordinates equal to the arithmetic means of the vertical boundaries of the strips, we should have had values 0·1 ft./sec./sec. lower throughout, giving a final speed lower by 0·1 ft./sec. $\times 20$ sec. $=2$ ft./sec., *i.e.* a final speed of 132 ft./sec. and definitely too low for upward convexity. (Actually the accelerations in feet/sec.^{2} are given by the expression $f=10-0{\cdot}025t^2$. And since $v=\int f dt$, $v=10t-0{\cdot}025t^3/3$, which, for $t=20$ sec., is $200-200/3$ $=133{\cdot}3$ ft./sec. And the distance traversed $=\int v dt=5t^2-t^4/480$ feet, which for $t=20$ gives 1666·7 feet.)

49. Two-dimensional Motion. If the motion of a point is not in a straight line but along a curved path, its displacement, velocity, and acceleration need vectors or components relative to fixed axes, to specify their directions. We need only consider motion in a curve lying wholly

in one plane, *i.e.* two-dimentional motion. The straight line joining two points P and P′ in Fig. 57 on the path of the moving point gives the magnitude and direction of the displacement in the corresponding interval of time, and the length of this displacement divided by the time taken gives the mean velocity during the interval. By considering two points closer together without limit to the smallness of the displacement and the time, we approach the actual velocity as a limiting value, in magnitude and direction, of a mean velocity for very small intervals of time and distance. The direction of the velocity at P would then be tangential to the path PP′ at P.

If at some instant the velocity of the moving point is given in magnitude and direction by the vector v, Fig. 58,

Fig. 57.

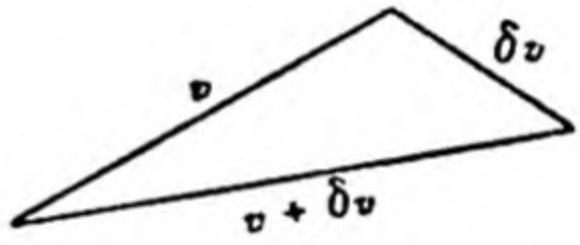

Fig. 58.

and after a small interval of time δt the velocity is given by $v+\delta v$, due to the geometrical or vector addition of the velocity δv, then the mean acceleration in the interval δt is

(Final velocity − initial velocity)/time)

or mean acceleration $=(v+\delta v-v)/\delta t=\delta v/\delta t$. (1)

or in the limit of diminishing displacements the actual acceleration is

$$f=dv/dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where f and v, being vector quantities, have direction as well as magnitude.

We may take equation (1) as giving the mean acceleration even when δv and δt are not small, but δv is a vector quantity having direction as well as magnitude.

It may be noticed that v and $v+\delta v$, being vector quantities, are not necessarily unequal in *magnitude* even when δv is not zero. The *speed* may be constant though the direction of motion may be changing. If we take δv as very small

and v equal to $v+\delta v$ in magnitude, the direction of δv is, in the limit, perpendicular to that of v, *i.e.* a point moving at a constant speed on a plane curve has an acceleration in the plane and perpendicular to its direction of motion.

The displacements, velocities, and accelerations may be resolved into rectangular components, referred to any pair of axes, for analytical solution of problems, but frequently vector explanations will serve our purpose in this book. We may, however, find it convenient to resolve into radial and tangential components accelerations of a point moving in a circular path.

EXAMPLE 1. A point moves due south at 6 feet per second, and 0·15 second later, it is moving at 8 feet per second in a direction 30 degrees south of west. Find the change of velocity in that time and the mean acceleration.

The directions of the final and initial velocities are shown by OR and OS in Fig. 59. If lengths OQ along OR

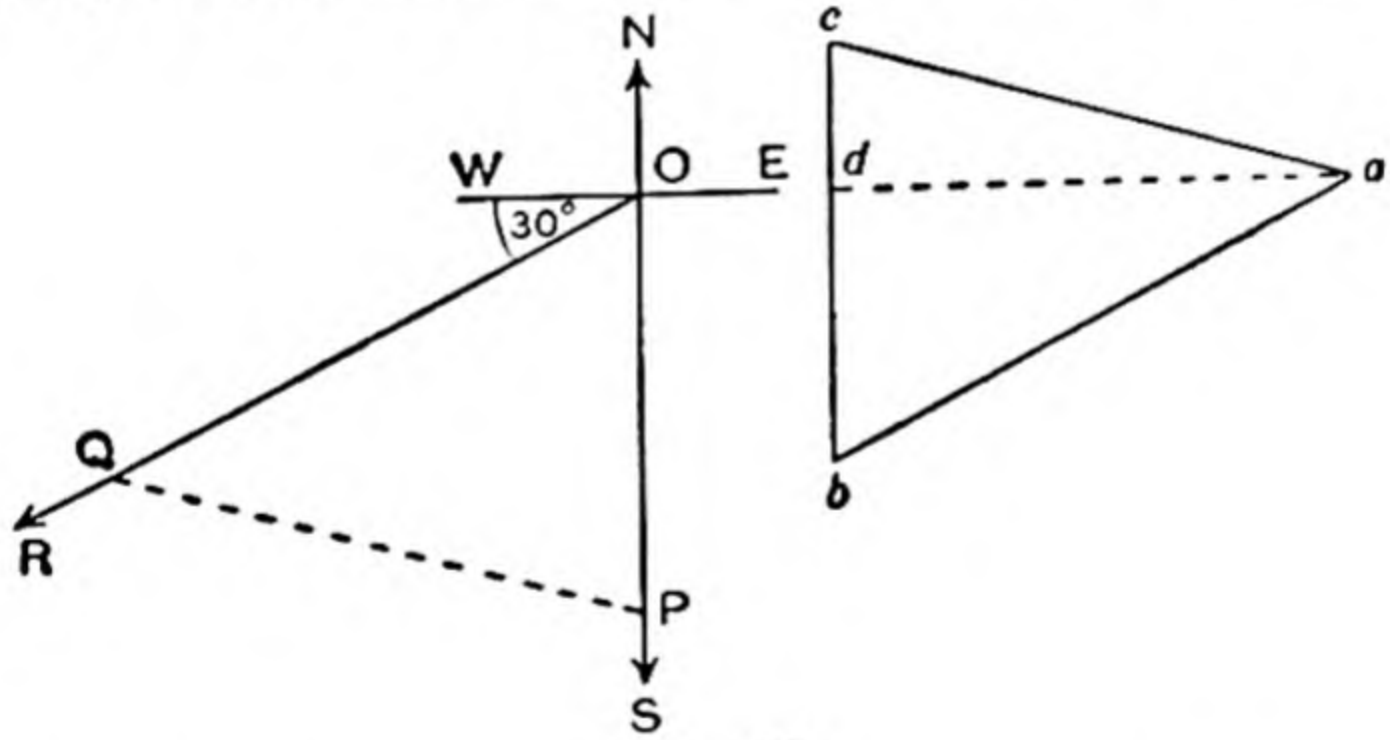

FIG. 59.

and OP along OS are set off proportional to 8 ft./sec. and 6 ft./sec. respectively, OPQ is a vector triangle; and evidently the addition to OP necessary to give the final velocity OQ is PQ, which scales 7·2 ft./sec. and is in a direction 16·1° north of west. A slightly different way of stating the matter is based on the vector triangle *abc*. The vector increase is the final velocity *ab* (=8 ft./sec.) minus the initial velocity *cb* (=6 ft./sec.) or

$$ab - cb = ab + bc = ac$$

which scales 7·2 ft./sec. and is inclined 16·1° to *ad*, *i.e.* to the west.

Instead of determining the magnitude by drawing to scale, *ac* can easily be calculated for $bd=8 \sin 30°=4$, $cd=6-4=2$, $da=4\sqrt{3}$, hence $ac=\sqrt{da^2+cd^2}=\sqrt{52}=7{\cdot}21$ ft./sec. Since the increase takes place in 0·15 sec. the rate of increase per second or the acceleration is (7·21 ft./sec.)÷0·15 sec. =48·1 ft./sec.^{2}.

50. Uniform Circular Motion. An important form of motion in a plane curve is motion at uniform speed (or uniform angular velocity) in the arc of a circle. Suppose a point describes, about a centre O (Fig. 60), a circle of radius *r* feet with uniform angular velocity ω radians per second.

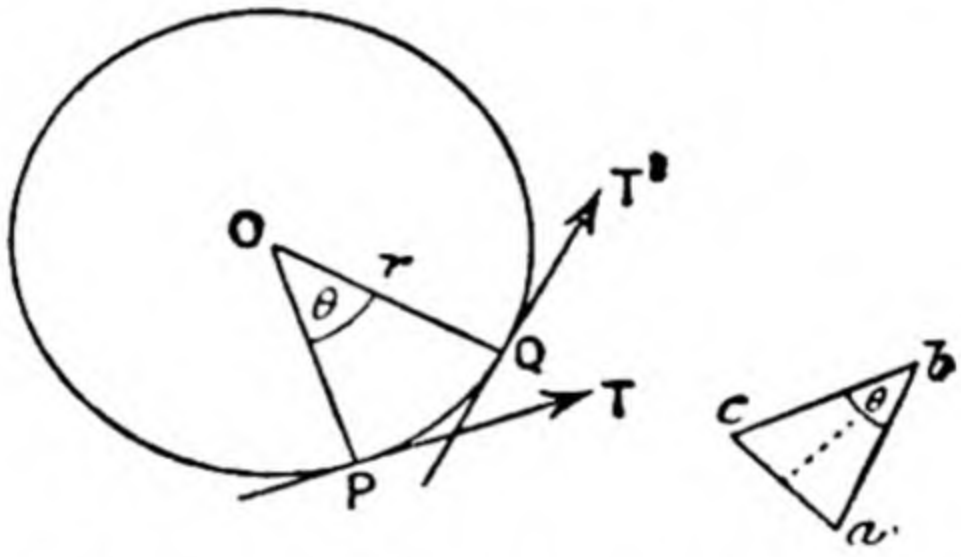

FIG. 60.—Uniform circular motion.

Then its velocity, *v*, at any instant is of magnitude ωr, and its direction is along the tangent to the circle from the point in the circumference which it occupies at that instant. Although its velocity is always of magnitude ωr, its direction changes. Consider the change in velocity between two points, P and Q, on its path at an angular distance θ apart (Fig. 60). Let the vector *cb*, parallel to the tangent PT, represent the linear velocity *v* at P, and let the vector *ab*, of equal length to *cb* and parallel to QT′, the tangent at Q, represent the linear velocity *v* at Q. Then, to find the change of velocity between P and Q, subtract the velocity at P from that at Q; in vectors,

$$ab-cb=ab+bc=ac.$$

Then the vector ac represents the change of velocity between the positions P and Q. Now, since $a\hat{b}c = P\hat{O}Q = \theta$, length $ac = 2ab \cdot \sin \frac{1}{2}\theta$, which represents $2v \sin \frac{1}{2}\theta$; and the time taken between the positions P and Q is θ/ω seconds.

Therefore the average change of velocity per second is

$$2v \sin \tfrac{1}{2}\theta \div (\theta/\omega) = \omega v \cdot \sin \tfrac{1}{2}\theta/(\tfrac{1}{2}\theta)$$

which is the mean or average acceleration. Now, suppose that Q is taken indefinitely close to P—that is, that the angle θ is indefinitely reduced; then the ratio $\sin \frac{1}{2}\theta/(\frac{1}{2}\theta)$ has a limiting value unity, and the mean change of velocity per second, or mean acceleration during an indefinitely short interval is ωv, or $\omega^2 r$ or v^2/r, since $v = \omega r$. This mean acceleration during an indefinitely reduced interval is the actual acceleration in feet per second per second. And as the angle θ is diminished indefinitely and Q thereby approaches P, the vector ab, remaining of the same length, approaches cb (a and c being always equidistant from b), and the angle $b\hat{c}a$ increases and approaches a right angle as θ approaches zero. Ultimately the acceleration ($\omega^2 r$) is perpendicular to PT, the tangent at P, *i.e.* it is towards O.

51. Relative Velocity. If two points A and B are moving in the same plane, the velocity of B relative to A is the velocity of B minus the velocity of A, the subtraction being, of course, a geometric or vector one and not arithmetic, except when A and B are moving in parallel straight lines.

Thus in Fig. 61 if two independent points A and B are moving in a fixed plane, which may be denoted by P, with velocities V_A and V_B respectively, and if these velocities are represented in magnitude and direction by the vectors pa and pb respectively (drawn from a common point p), then the velocity of B relative to A is ab. For, in vectors,

$$pb = pa + ab$$

or,

$$pb - pa = ab.$$

That is, the velocity of B minus the velocity of A.

If the whole plane P and the points moving in it had impressed upon them a velocity $-V_A$, the point A would be at rest; and the velocity of B relative to A (or as apparent from A), would be unchanged and it would evidently be

V_B plus the velocity $-V_A$, or vectorially $pb - pa$, which is ab.

The notation of the vectors is such that the second letter (b) gives the point to which the velocity refers, and the first letter (a) gives the point or body to which the velocity is relative. Thus pb is the velocity of B relative to P (taken as fixed), while ab is the velocity of B relative to A. Similarly, ba is the velocity of A relative to B and pa the velocity of A relative to P. We shall adopt this convention in what follows and, generally, fixed points or bodies will be denoted by P.

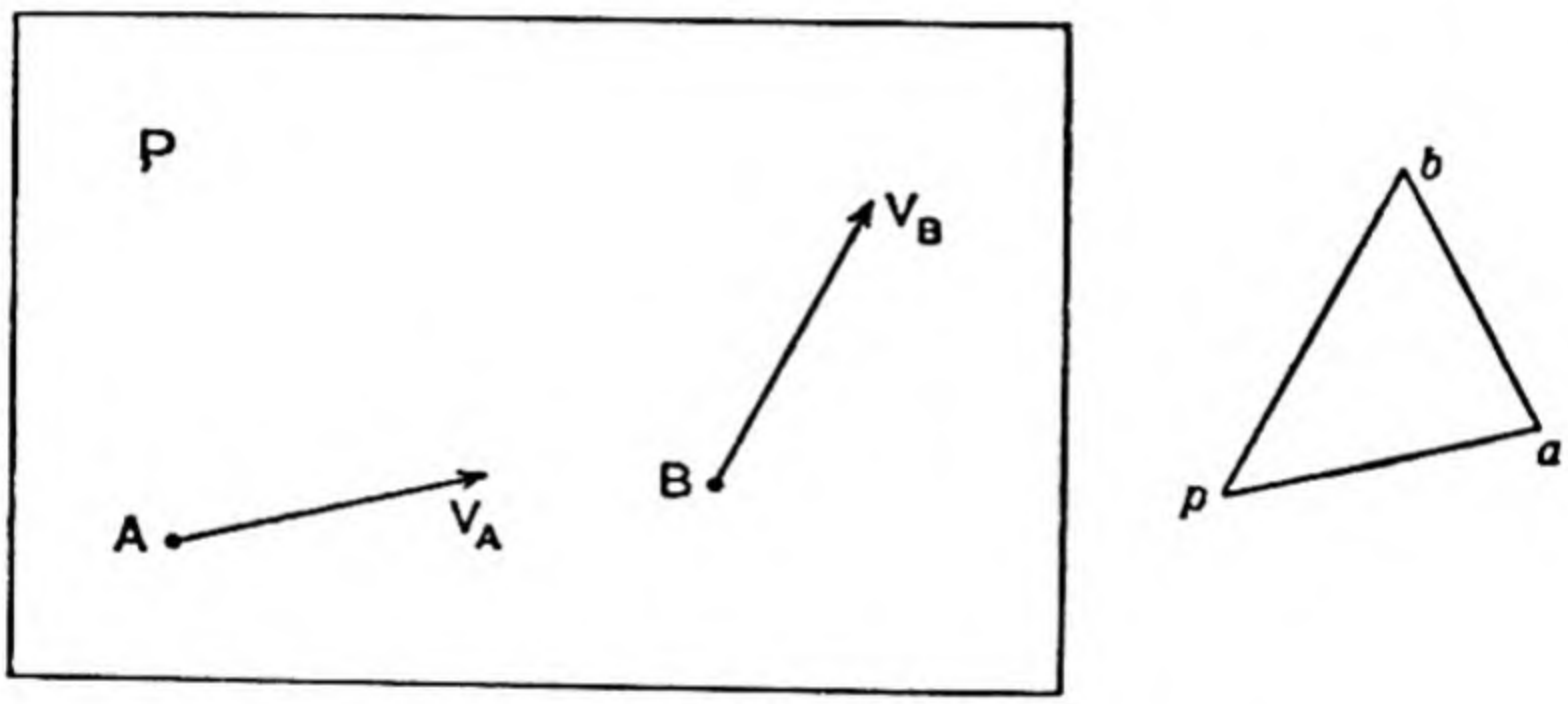

FIG. 61.

EXAMPLE. Two straight railway lines cross: on the first, a train 10 miles away from the crossing, and due west of it, is approaching at 50 miles per hour; on the second, a train 20 miles away, and 15° E. of N., is approaching at 40 miles per hour. How far from the crossing will each train be when they are nearest together, and how long after they occupied the above positions?

First set out the two lines at the proper angles, as in the left side of Fig. 62, and mark the positions A and B of the first and second trains respectively. Now, since the second train B is coming *from* 15° E. of N., the first train A has, *relative to the second*, a component velocity of 40 miles per hour in a direction E. of N., in addition to a component 50 miles per hour due east. The relative velocity is therefore found by adding the vectors pq 50

miles per hour east, and *qr* 40 miles per hour 15° E. of N., giving the vector *pr*, which scales 72 miles per hour, and has a direction 57½° E. of N. Now draw from A a line AD parallel to *pr*. This gives the positions of A relative to B (regarded as stationary). The nearest approach is evidently a distance BD, where BD is perpendicular to AD. The distance moved by A relative to B is then AD, which scales 23·2 miles (the trains being then a distance BD, which scales

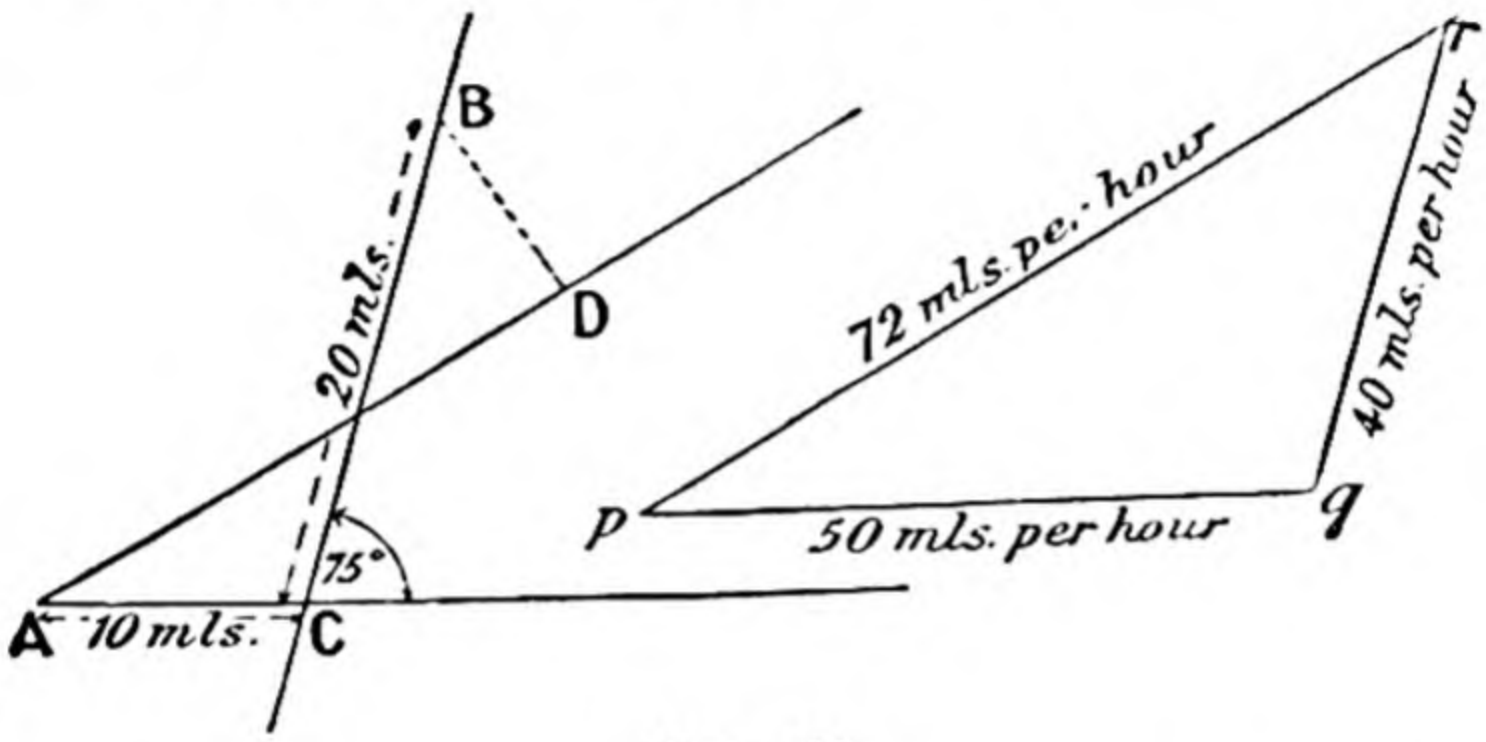

FIG. 62.

8·12 miles, apart). The time taken to travel relatively 23·2 miles at 72 miles per hour is 23·2/72 hour = 0·322 hour.

Hence A will have travelled 50 × 0·322 or 16·1 miles
and B „ „ 40 × 0·322 or 12·9 „
A will then be 6·1 miles past the crossing, and
B „ „ „ 7·1 „ short of the crossing.

52. Motion of a Link. In the preceding article (51) we considered the velocities of two points with entirely independent motions in the same plane. In the moving parts of a machine we often require to study the motion of two points on one rigid piece of the mechanism which we may conveniently call a link. The motion of two such points may be very different but not independent, and when the velocity of one is specified we may be able to find that of the other from sufficient data. The fundamental fact is that the two points remain at a fixed distance apart.

Hence the velocity of each of the two points, relative to the other, has no component along the line joining them and can only be perpendicular to it. In other words, if two points A and B are on a link moving in the plane of AB, the velocity of A relative to B (and of B relative to A) is perpendicular to AB.

Suppose the velocity of B, Fig. 63, is V_B, of given magnitude in the direction shown in the position diagram, and the velocity of A is in the direction shown in the position diagram. From any point p, set off a vector pb to represent V_B in magnitude and direction, and set off the indefinite line pa in the given direction of V_A. The length pa is then determined from the fact that the velocity of B is equal to the velocity of A plus that of B relative to A, and this relative velocity is perpendicular to AB. Hence if we

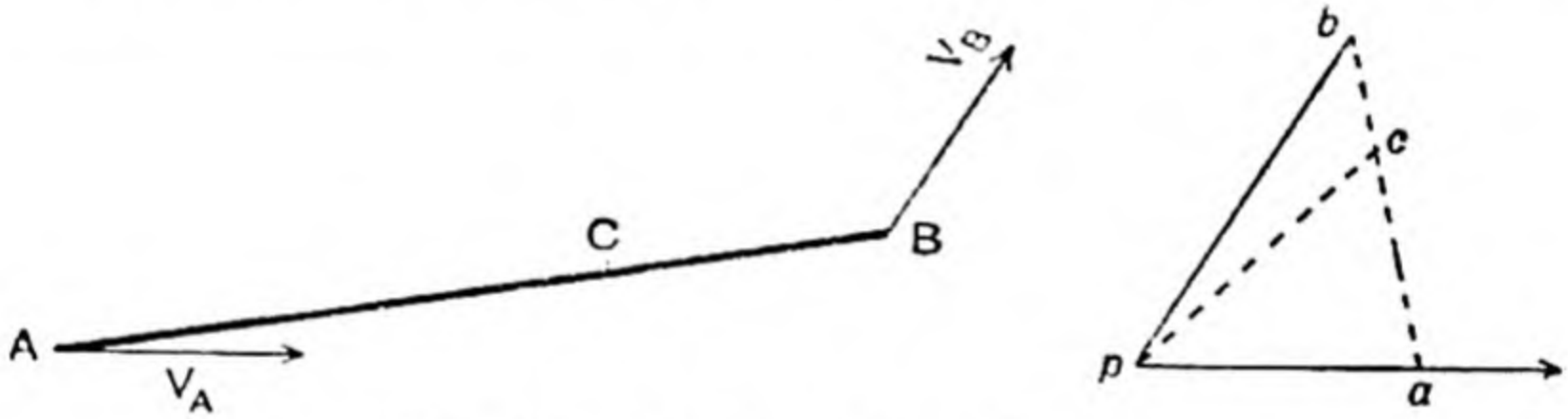

FIG. 63.—Plane motion of a link.

draw from b a line bc perpendicular to AB to meet the indefinite line from p parallel to V_A in a, we complete the vector triangle pba such that

$$pb = pa + ab \quad \text{or}$$

Velocity of B = velocity of A + velocity of B relative to A.

Thus pa, the velocity of A, is completely determined. We have determined not only the velocity of A but also that of B relative to A (by ab) and A relative to B (by ba), both perpendicular to AB. This velocity of B relative to A, divided by AB, gives the angular velocity of the link AB. It is here (in Fig. 63) obviously counter-clockwise.

If (in addition to V_B) the magnitude only of V_A (instead of its direction) were known, by drawing from b a line perpendicular to AB and drawing an arc from the centre p

with radius *pa* to represent the magnitude of V_A, we should determine the intersection *a* and the triangle *pba* and the direction of V_A (subject to the facts (1) that the magnitude of V_A must be at least equal to the perpendicular distance of *p* from *pa*, *i.e.* to its component in the direction AB, for the component of V_A must be equal to the component of V_B in this direction); and (2) that there will be two intersections and therefore two possible directions for V_A.

The velocity of a point C on AB is found by dividing *ab* at *c* in the ratio of AC to CB, *i.e.* so that *ac*/*cb* = AC/CB, for the point C must have a velocity having the same component as A and B in the direction AB and relative to A a velocity AC/AB times that of B about A. Thus the vector *pc* gives the velocity of C, for its components obviously comply with these requirements. If the point C were not in the line AB but on the same rigid link as AB, the points *c* could be found by drawing on *ab* a triangle *abc* similar to the triangle ABC.

Instantaneous Centre. Any motion of a link in its own plane may be regarded as a rotation about a point in he plane. This centre of rotation, for a finite movement of the link, is easily found, but generally differs for different amounts of displacement. For an indefinitely small displacement, such a centre is called the Instantaneous Centre of Rotation. This centre provides us with an approach to the velocity of all points and to the velocity diagram alternative to that already employed and illustrated in Fig. 63.

The link AB of Fig. 63 is again shown in Fig. 64. If A is moving in the direction V_A, any centre about which it may be instantaneously swinging must be in the line AI drawn perpendicular to the direction of V_A. And similarly it must be in the line IB perpendicular to the direction of V_B. Hence the instantaneous centre of rotation must be at I, the point of intersection of the lines IA and IB respectively perpendicular to V_A and V_B. Then the motion of any other point C in the link must be perpendicular to a line joining I to C, *i.e.* to IC. The velocity of A must be $V_B \times$ IA/IB (and of C is $V_B \times$ IC/IB). The triangle IAB is similar to *pab* but rotated through a right angle (clock-wise), where *pab* is the vector velocity diagram drawn as in

Fig. 63.

If velocities of A or C or other points are required for several positions of the link, the vector velocity diagrams such as *pab*, drawn all to the same scale are more convenient than triangles such as IAB, to scales which vary with the positions of I and the lengths IB, representing what may be a velocity of constant magnitude (even though varying in direction). Sometimes, however, the vector velocity diagrams such as *pab* are drawn in positions corresponding to IAB, *i.e.* with lines perpendicular to the directions of the velocities which they represent. When this is done, the velocity of any point, such as C, is given in direction by the motion of *c* in a rotation of the vector diagram about *p* and in magnitude by that of the velocity

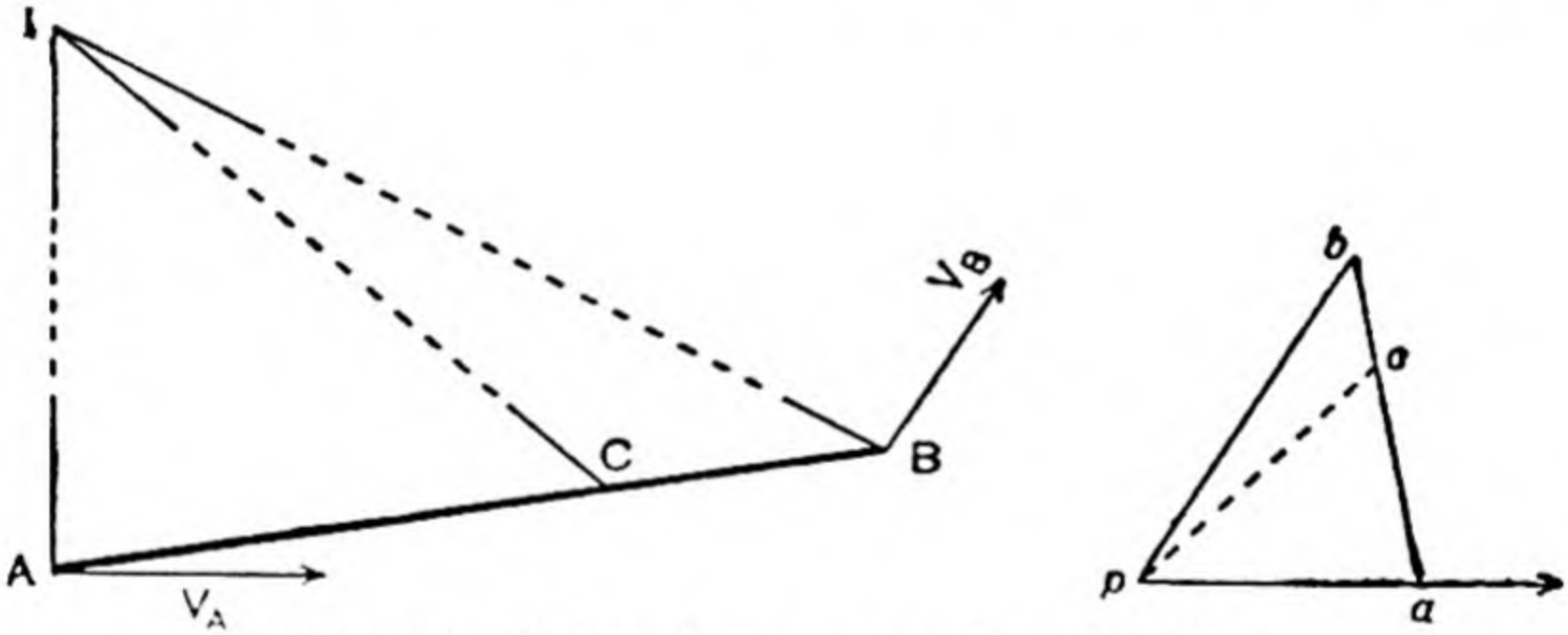

FIG. 64.—Instantaneous centre of motion.

of *c* when the vector diagram is rotated about *p* at a proper angular velocity, readily found by the angular speed required to give the speed V_B at the radius *pb*. At this angular speed, not only will the vector *pc* give the speed of any point C, but rotation of the vector diagram at the same angular speed about any other point will, by its radius vector from that centre, give the magnitude of the velocity of any other point in the link relative to the corresponding point on the link, *e.g.* the velocity of B relative to C is given by that of *b* when *pab* is rotating about *c* with the previously specified angular velocity.

53. Motion of Paired Links. The determination of the motion of all points in a link in the previous article (52) leads to the determination of the motion of a second link

paired with it, *i.e.* a link which has completely controlled motion relative to it and in contact with it by a union such as a pin joint or a sliding surface. This extension of velocity determination may be illustrated by the mechanism of a simple reciprocating engine shown diagrammatically in Fig. 65. The link AB is the connecting rod and BC is the crank, joined or paired to the rod at B,

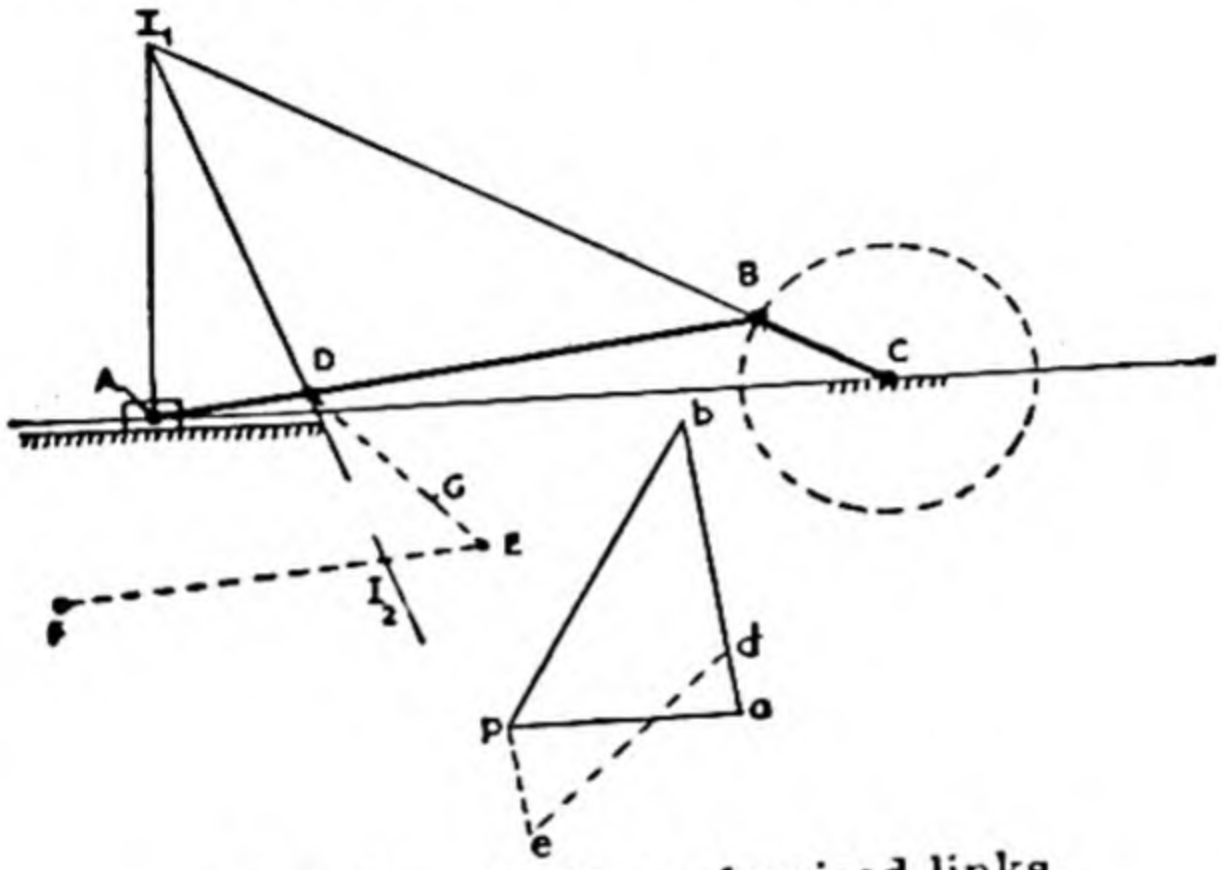

FIG. 65.—Motion of paired links.

while at A is a pin connecting the rod to the crosshead which moves horizontally on a fixed slide bar or bed. The vector diagram *pab* is drawn as before and the velocity of any point D is found by dividing *ba* at *d* as BA is divided at D, or by drawing *pd* perpendicular to I_1D where I_1 is the instantaneous centre for the connecting rod AB. Now suppose a link DE is pin-jointed to AB at D and the end E is constrained to move in a circular arc about a fixed point F, by joining it at E to a radius bar FE pin-jointed to and moving about F. The motion of E is determined by the facts that it is perpendicular to FE, and relative to D, it is perpendicular to DE. A vector *pe* perpendicular to FE will meet a line *de* perpendicular to DE in *e* and the velocity of E is represented by the vector *pe* in magnitude and direction. The motion of any point such as G on DE would be found by dividing

ed at a point *g* (not shown) such that *eg*/*ed* = EG/ED,

or by drawing from p a line pg perpendicular to I_2G where I_2 is the instantaneous centre of motion of the link DE (FE is perpendicular to the direction of motion of E and DI_2 is perpendicular to the direction of motion of D. These two lines meet in I_2 which is therefore the instantaneous centre of the link DE).

It has been assumed that the velocities of the points in the mechanism are being determined by drawing velocity diagrams to scale from which the velocities may be measured. We may, however, state a velocity in terms of the dimensions in the position or space diagram and the given velocity of a point such as B, say, V_B. Thus

$$V_A = \text{velocity of A} = V_B \times I_1A/I_1B$$
$$V_D = \text{velocity of D} = V_B \times I_1D/I_1B$$
$$\text{Velocity of G} = V_B \times (I_1D/I_1B) \times (I_2G/I_2D)$$

EXAMPLE. A and B are two axes at the same level on a fixed base. Cranks BC and AD can turn about B and A respectively. They are connected by a coupling bar DC hinged to AD at D and to BC at C. If AB=25 inches, BC=15 inches, AD=10 inches, and DC=13 inches, find

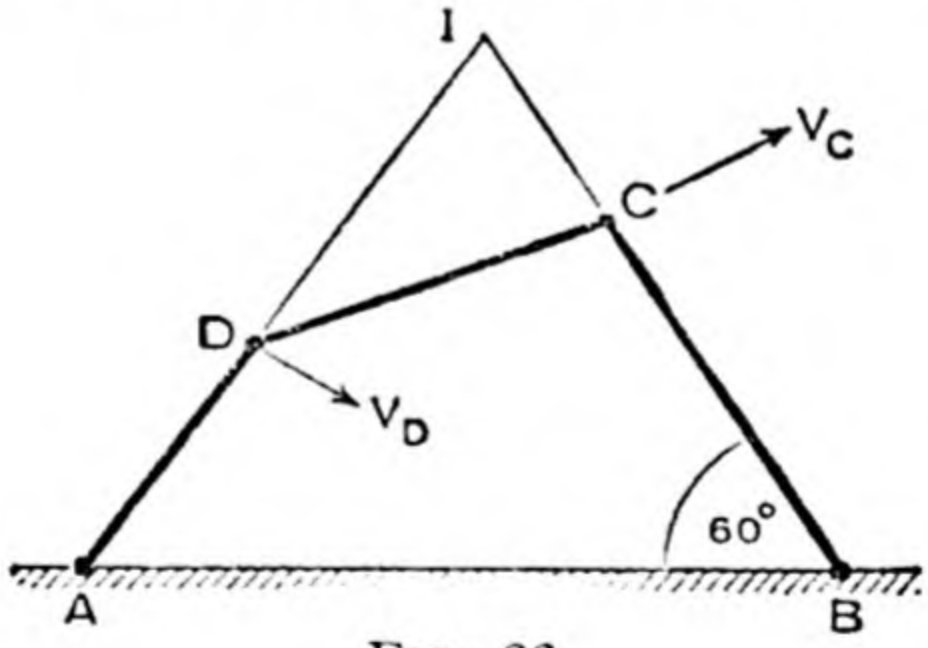

FIG. 66.

the magnitude of the velocity of D in terms of the velocity V_C of C when BC is inclined 60° to AB, D and C being on the same side of AB Find also the ratio of the angular velocity of the crank AD to that of BC.

The position diagram is shown in Fig. 66. BC and AD produced meet in I, which is the instantaneous centre of

motion of the link DC. Since this link may be considered as rotating instantaneously about I,

$$V_D/V_C = ID/IC$$

which, by measurement of the lengths ID and IC and division, gives 1·78 or $V_D = 1{\cdot}78\ V_C$.

The angular velocity of BC about B is V_C/CB, while that of AD about A is V_D/AD. Hence the angular velocity of AD divided by that of BC is

$$(V_D/AD) \div (V_C/BC) \text{ or } (V_D/V_C) \times (BC/AD)$$

which is $1{\cdot}78 \times 1{\cdot}5$ or 2·67, which is the ratio of the angular velocity of the crank AD to that of the crank BC. (It is easy to show that this ratio is also the ratio of the distances of B and A from the point of intersection of DC with AB, both produced.)

54. Mechanisms, Machines and Velocity Ratios. Two links (kinematically) paired one with the other have their relative motions controlled; and when several links are connected together so that if one is fixed, the motion of all other parts is so completely controlled or constrained as to be quite definite, the combination is called a mechanism. Mechanisms can transmit and modify motions; we think of a mechanism as transmitting motion without reference to the forces or absolute speeds. When a moving mechanism transmits considerable forces and therefore appreciable power, it is commonly called a machine. Without attempting to state exact definitions, which are apt to be confusing, the foregoing will sufficiently explain the terms *mechanism* and *machine*.

The velocity (or movement) ratio between two points in a machine or mechanism may be found by various means, and we have employed constant and varying velocity ratios in Chapter III. Variable velocity ratios may often conveniently be found from vector velocity diagrams such as are illustrated in Art. 53. Moreover, we can employ the term velocity ratio, not only as the ratio of the velocities of two points in their actual directions but in directions of forces exerted at the points. Thus in the toggle joint (Art. 46, Fig. 51), we determined the velocity ratio of the effort P to that of the load W where the direction of P

was inclined to the direction of motion (and velocity) of the point to which P was applied. It is very easy to project a vector of a velocity diagram in any required direction to get a particular velocity ratio.

55. Applications of Velocity Diagrams. Vector velocity diagrams may be used for ascertaining velocities of parts of machines in various positions of the component links of the mechanism. From such determinations of velocities, velocity-time graphs may be plotted and the acceleration may be determined for points having rectilinear motion (as in Art. 48). From such kinematical considerations, with data relating to the masses moved, the forces involved in fast-running machinery may be deduced by the laws of dynamics (Chap. V).

But we have also seen from the Principle of Work for the ideal machine (Art. 46) that a knowledge of the velocity ratio gives also the force ratio in a mechanism and vice versa, the force ratio being the reciprocal of the velocity ratio when friction is negligible. Thus an unknown force acting on a machine may be determined from a velocity diagram. And this use of a velocity diagram may be extended to structures by conceiving of a small movement of a point in a jointed structure, arising either from elastic yielding of the members of the structure or from the imaginary removal of a link or part which would allow movement: in other words, would convert the rigid structure into a mechanism. The subject of this application may be amplified in the further study of Theory of Machines and Structures.[1]

EXAMPLE. Find the velocity ratio of the points X and Y and of the forces P and W in the toggle joint of Example 2, Art. 46 (Fig. 51).

The position diagram is shown on the left side of Fig. 67. The point Z being fixed, the point X can only move at right angles to ZX. Draw zx to represent its velocity to scale. Relative to Y, X must move perpendicular to YX; yx is drawn perpendicular to YX, and since Y moves

[1] See "Velocity Diagrams," by S. H. Stelfox in *Engineering*, December 31, 1937.

vertically, zy drawn vertically from z to meet the line xy in y, completes the velocity diagram.

The velocity ratio

$$=\text{Speed of X/speed of Y}=zx/zy$$
$$=zx/(2zx \sin\theta)=1/(2\sin\theta)$$

But the ratio of the horizontal speed of P to the vertical speed of W is hx/zy or $\frac{1}{2}\cot\theta$, which was the ratio W/P in Example 2, Art. 46. Thus in terms of the instantaneous rate of work per second (or power), the principle of work for this mechanism is

$$W \times zy = P \times xh, \text{ or } W/P = hx/zy = \tfrac{1}{2}\cot\theta$$

It will be noticed that the vector velocity diagram zxy of Fig. 67 is precisely similar to the vector force diagram of

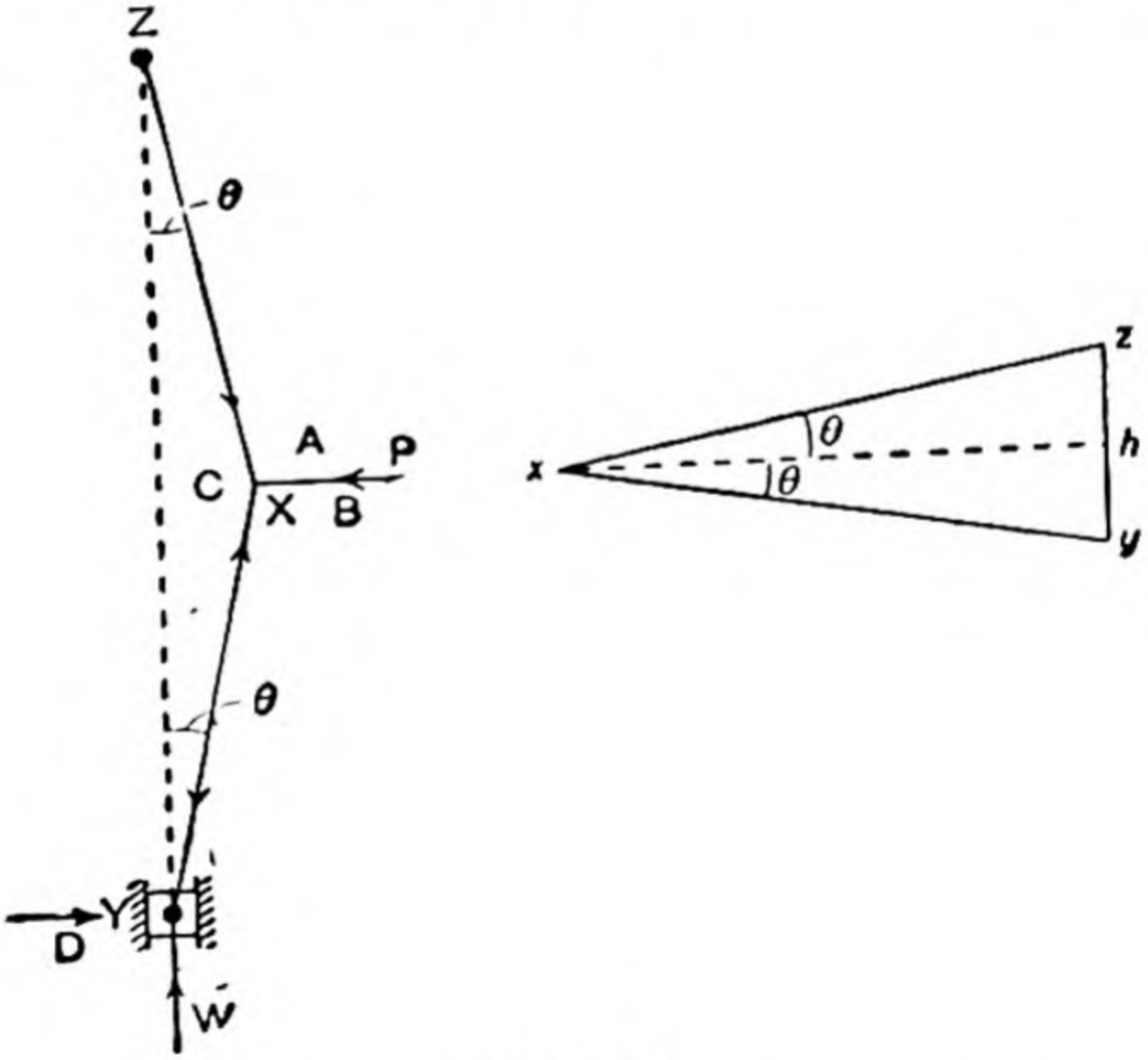

FIG. 67.

Fig. 51, and thus the velocity diagram and the force diagram are alternatives. If the velocity diagram is drawn with

sides perpendicular to, instead of parallel to, the velocities, the vector diagrams of velocities and forces are even more obviously interchangeable. They represent two aspects of the same facts already referred to in Art. 46.

EXAMPLES IV

Examples Nos. 1, 2, 3, *and* 4 *are to be worked graphically.*

1. A train starting from rest covers the distances s feet in the times t seconds as follows :—

t .	0	5	11	18	22	27	31	38	46	50
s .	0	10	54	170	260	390	450	504	550	570

Find the mean velocity during the first 10 seconds, during the first 30 seconds, and during the first 50 seconds. Also find approximately the actual velocity after 5, 15, 25, 35, and 45 seconds from starting-point, and plot a graph showing the velocities at all times.

2. Using the graph of velocities from Example 1, find the acceleration every 5 seconds, and draw the curve of accelerations during the first 40 seconds.

3. A train travelling at 30 miles per hour has steam shut off and brakes applied ; its speed after t seconds is shown in the following table :—

t . . .	0	4	12	20	26	35	42	50
v, miles per hour . .	30·0	26·0	21·5	16·7	14·0	10·4	7·7	4·8

Find the retardation in foot and second units at 5-second intervals throughout the whole period, and show the retardation by means of a graph. Read off from the graph the retardation after 7 seconds and after 32 seconds. What distance does the train cover in the first 30 seconds after the brakes are applied ?

4. A body is lifted vertically from rest, and is known to have the following accelerations f in feet per second per second after times t seconds :—

t .	0	0·8	1·9	3·0	3·39	4·8	6·0	6·8	8·0	8·8
f .	3·0	2·9	2·85	2·60	2·20	1·75	1·36	1·20	1·04	0·97

Find its velocity after each second, and plot a graph showing its velocity at all times until it has been in motion 8 seconds. How far has it moved in the 8 seconds, and how long does it take to rise 12 feet?

5. A wheel 8 feet diameter is rotating in fixed bearings at 6 radians per second and has an angular acceleration of 2 radians per second per second. Find the initial velocity of a point on the rim vertically over the axis of rotation and the position and velocity of the same point after 0·2 second. Find also the magnitude and direction of the mean acceleration during that time.

6. A wheel of radius r, rotates at a speed of ω radians per second and has an angular acceleration of α radians per second per second. Find the magnitude and direction of the acceleration of a point on the rim.

7. An engine crank, one foot long, turns at 6 radians per second, being driven by a piston rod, crosshead, and a connecting rod 4 feet long. When the crankshaft is at an angle of 30° to the centre line through the crosshead and crankshaft, find (*a*) the speed of the crosshead, and (*b*) the angular velocity of swing of the connecting rod. What would be the torque on the crankshaft per 1000 pounds thrust of the piston? (Use the principle of work.)

CHAPTER V

PRINCIPLES OF DYNAMICS

56. Laws of Motion. The three so-called laws of motion may be regarded as assumptions sufficiently verified for practical purposes by experience. They refer primarily to the behaviour of a particle, a portion of matter of indefinitely small size, but having appreciable mass. But we may by other justifiable assumptions relate them to the behaviour of solids or rigid bodies of finite size. In applied mechanics we are concerned with the application of the theoretical principles rather than with their philosophical justification. We shall assume that the forces and torques necessary to give a rigid body linear and angular acceleration are the same as those required to give these accelerations to the groups of particles of which the rigid body may consist. Such an assumption is amply justified by experience and does not involve any doubtful hypothesis.

57. First Law. (*a*) *Linear Motion.* A body continues in its state of rest or of motion in a straight line with constant velocity except in so far as it is compelled by external force to change that state.

(*b*) *Angular Motion.* A body continues in its state of rotation (or of rest) about any axis with constant angular velocity except in so far as it is compelled by external torque to change that state.

This law is only a particular case of the second law when the rate of change of momentum (or of angular momentum) is zero. We may notice a demonstration of its truth (for linear motion) when a stone tied to a string is whirled in a circle. If the stone is suddenly released from the controlling force exercised by the string it moves in a straight line tangential to the circle (*not* radially) although this motion at once begins to be modified by the gravitational pull exerted on the stone. Similarly, drops of water leaving the tyre of a car start their free paths tangentially to the tyre.

58. Second Law. (*a*) *Linear Motion.* The rate of change of momentum of a body is proportional to the unbalanced force exerted on it and takes place in the direction of the straight line in which the force acts.

(*b*) *Angular Motion.* The rate of change of angular momentum (or moment of momentum) of a body is proportional to the torque exerted and is about the axis of the torque.

If the mass m or W/g of the body is constant and v is the variable velocity, the momentum is mv and the rate of change of momentum is m times the rate of change of v, which is $m \times$acceleration, and is equal to P the unbalanced or resultant force exerted. Thus if f is the acceleration in the direction of P

$$P = d(mv)/dt = m\,dv/dt = mf = Wf/g \quad . \quad . \quad (1)$$

Similarly, if the moment of inertia I (or mk^2) of a body, of radius of gyration k, is constant, and ω is the variable angular velocity, the rate of change of angular momentum $(mk^2\omega)$ is $mk^2 \times$(rate change of ω) or $mk^2 \times$angular acceleration, which is equal to the unbalanced or effective torque T or

$$\left.\begin{aligned} T &= I \times \text{angular acceleration} \\ &= mk^2 \times \text{angular acceleration} \\ &= Wk^2/g \times \text{angular acceleration} \end{aligned}\right\} \quad . \quad . \quad (2)$$

$$T = d(mk^2\omega)/dt = mk^2\,d\omega/dt = I\,d\omega/dt = I\alpha \quad . \quad (3)$$

where α is the angular acceleration produced about the axis of T.

Impulse. The second law is stated in terms of a *rate of change* (*a*) of momentum or (*b*), in the case of angular motion, of moment of momentum. Summing the effects of constant or varying forces or torques over a definite time, the corresponding statements become for a rigid body of constant mass and shape

$$\left.\begin{aligned} (a)\ \text{Impulse} = \int P \,.\, dt &= \text{total change of momentum} \\ &= m \times \text{total change of velocity} \end{aligned}\right\} \quad . \quad (4)$$

$$\left.\begin{aligned} (b)\ \text{Angular impulse} \\ = \int T \,.\, dt &= \text{total change of angular momentum} \\ &= I \times \text{total change of angular velocity} \end{aligned}\right\} \quad . \quad (5)$$

In the case of a constant force, the impulse is $P \times t$ where t or $\int dt$ is the time for which the force acts. So that (4) becomes

$$Pt = m(v_2 - v_1) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where v_2 is the final and v_1 the initial velocity of mass m.

If we divide by t, we get

$$P = m(v_2 - v_1)/t = mf \quad . \quad . \quad . \quad . \quad (7)$$

which is the statement of the second law as in (1).

Similarly, if we differentiated (4), we should get

$$P = mdv/dt = mf \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Thus (1) and (4) are different statements of the same law, (1) being a rate of change, (4) and (6) total changes.

When all parts of a rigid body have the same rectilinear motion and therefore the same acceleration, the resultant of all the forces necessary to accelerate the parts will have a resultant through the centre of gravity of the body. For each separate force is proportional to the mass of the part it accelerates, just as each gravitational pull on a part is proportional to the mass of the part, and the process of finding the centre of the accelerating forces is exactly similar to that of finding the centre of the gravitational forces. If the resultant of certain external forces on a body acts other than through the centre of gravity, it will either call into existence other reactive forces from some supporting agent (such as a rail) or cause the body to rotate, as well as to accelerate in the direction of the force. And the rotary effect will be the same as if the particles constituting the rigid body were concentrated at the radius of gyration about the axis of the resultant couple formed by the external forces.

D'Alembert's Principle. In solving problems relating to the forces and moments accelerating a rigid body, the practice is sometimes adopted of introducing fictitious forces equal and opposite to the resultant of the external forces and so placed as to have moments equal and opposite to the resultant moments of the external forces. These fictitious forces are therefore also equal (but opposite) to the product of the mass of the body multiplied by the acceleration of the centre of gravity, and have a moment

equal (but opposite) to the product of the mass of the body multiplied by the square of the radius of gyration and by the angular acceleration of the body about the axis of the resultant couple formed by the external forces. The effect of these fictitious additions is to reduce a problem on motion to one in statics. In some instances this may appear slightly to simplify the routine part of working out a problem. But its general adoption is to be deprecated as obscuring the physical realities and is particularly undesirable for the learner. The second law is best appreciated by working chosen examples. In those which follow, different points including this form of d'Alembert's Principle will be illustrated.

EXAMPLE 1. The piston, piston rod and crosshead of an engine weigh 80 lb. and when in line with the connecting rod have an acceleration of 350 ft./sec.^{2}. If the thrust of the steam on the piston is 4000 lb., find the thrust exerted by the crosshead on the connecting rod.

The force which accelerates the piston, etc., is

$$f.W/g = 350\ \frac{\text{ft.}}{\text{sec.}^2} \times 80\ \text{lb.} \times \frac{1}{32 \cdot 2} \cdot \frac{\text{sec.}^2}{\text{ft.}} = 869\ \text{lb. (force)}$$

The force transmitted to the connecting rod is therefore 4000 lb. $-$869 lb. $=$3131 lb. The forces are shown in Fig. 68. Those thrusting inward are the external forces on

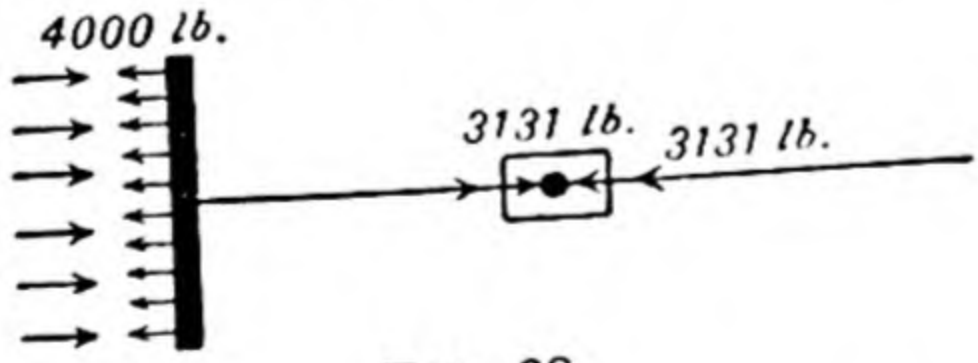

FIG. 68.

the piston, piston rod, and crosshead, which form a single rigid body, while those thrusting outward are the forces exerted by this body. The reciprocating body has exerted upon it an unbalanced, effective or accelerating force of 4000 $-$ 3131 $=$ 869 lb.

EXAMPLE 2. A locomotive exerts a constant draw-bar pull of 4 tons on a train weighing 200 tons when hauling

it up an incline of 1 in 120. If the resistance of the rails, etc., amounts to 10 lb. per ton, how long will the train take to attain a velocity of 25 miles per hour from rest, and how far will it have moved ?

The forces resisting motion are

(*a*) Force of gravity $=\frac{1}{120}$ of 200 tons

$$=\frac{200\times2240}{120}=3733 \text{ lb.}$$

(*b*) Resistance at 10 lb. per ton $=200\times10=2000$ lb.

Total . 5733 „

a pull which suffices for constant speed.

The draw-bar pull is $4\times2240=8960$ lb.; hence the net force available for acceleration $=8960-5733=3227$ lb.

Let t be the time required; then the impulse is 3227 lb. (force) $\times t$. The velocity gained is $25\times5280/(60\times60)=36{\cdot}7$ ft./sec. Gain of momentum is

$$mv=\frac{W}{g}\,.\,v=\frac{200\times2240 \text{ lb.}\times\text{sec.}^2}{32{\cdot}2 \quad \text{ft.}}\times36{\cdot}7\ \frac{\text{ft.}}{\text{sec.}}$$

Equating the impulse to the gain of momentum

$$3227 \text{ lb. (force)}\times t=\frac{200\times2240}{32{\cdot}2}\times36{\cdot}7 \text{ lb.(force)}\times\text{sec.}$$

$$t=158{\cdot}4 \text{ sec., or 2 minutes, 38 seconds.}$$

Since the acceleration has been uniform from rest, the average speed is half the maximum and the distance travelled will be

$$\tfrac{1}{2}\times36{\cdot}7\text{ft./sec.}\times158{\cdot}4 \text{ sec.}=2903 \text{ ft.}$$

EXAMPLE 3. A car weighing 12 tons starts from rest, and has a constant resistance of 500 lb. The tractive force P, in lb., on the car after t seconds is as follows :—

t .	0	2	5	8	11	13	16	19	20
P .	1280	1270	1220	1110	905	800	720	670	660

Find the velocity of the car after 20 seconds from rest, and show how to find the velocity at any time after starting, and to find the distance covered up to any time.

Plot the graph of P and t, as in Fig. 69, and read off the

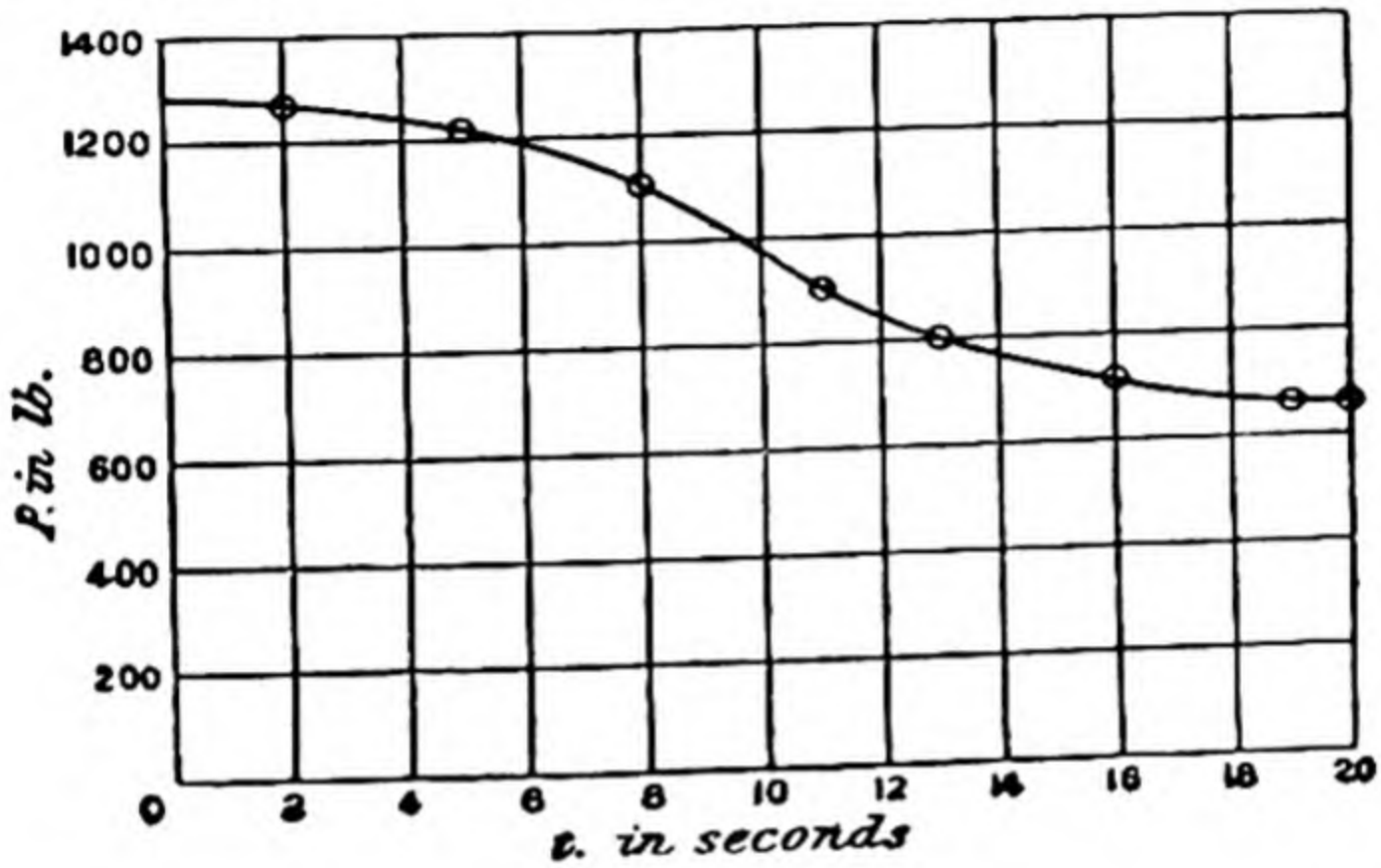

FIG. 69.

force, say every 4 seconds, starting from $t=2$ sec., and subtract the 500 lb. resistance from each as follows :—

t . .	2	6	10	14	18
P lb. .	1270	1190	980	760	680
P − 500 .	770	690	480	260	180

The mean accelerating force in the first 4 seconds is approximately 770 lb., and therefore the impulse is 770×4 lb.(force) $\times$ sec., which is also the gain of momentum.

The mass of the car

$$= \frac{12 \times 2240 \text{ lb.(force)} \times \text{sec.}^2}{32{\cdot}2 \text{ ft.}} = 835 \frac{\text{lb.(force)} \times \text{sec.}^2}{\text{ft.}}$$

The velocity after 4 sec.

$$= \frac{\text{momentum}}{\text{mass}} = \frac{770 \times 4 \text{ lb.} \times \text{sec.} \times \text{ft.}}{835 \text{ lb.} \times \text{sec.}^2} = 3{\cdot}69 \text{ ft./sec.}$$

Similarly, finding the momentum and gain of velocity in each 4 seconds, we have

t	0	4	8	12	16	20
Gain of momentum in 4 seconds .	0	3080	2760	1920	1040	720
Momentum . .	0	3080	5840	7760	8800	9520
Velocity, feet per second . .	0	3·69	7·00	9·29	10·54	11·40

After 20 seconds, the velocity is approximately 11·40 feet per second. The velocity after any time may be obtained approximately by plotting a graph of velocities and times from the values obtained, and reading intermediate values. More points on the velocity-time graphs may be found if greater accuracy be desired.

The space described is represented by the area under the velocity-time graph, and may be found as in Art. 48.

EXAMPLE 4. A motor, in starting, exerts on the shaft for one minute a constant turning moment of 1000 lb.-feet, and there is a uniform moment resisting motion, of 800 lb.-feet. A flywheel on the shaft has a radius of gyration of 5 feet and weighs 2000 lb. Neglecting the inertia of all parts except the flywheel, what speed will the shaft attain in one minute?

(1) Considering the rate of change of angular momentum—

The effective turning moment is $1000-800=200$ lb.-feet.

The moment of inertia of the flywheel

$$=\frac{2000}{32\cdot2}\times5^2\ \frac{\text{lb.}\times\text{ft.}^2\times\text{sec.}^2}{\text{ft.}}=1553\ \text{lb.-ft.-sec}^2.$$

Hence if α = angular acceleration in radians per second per second

$$200\ \text{lb.-ft.}=1553\ \text{lb.-ft.-sec.}^2\times\alpha$$

$$\alpha=200/1553\ \text{radian/sec.}^2=0\cdot1288\ \text{radian/sec.}^2$$

And the angular velocity attained in one minute

$=60$ sec. $\times 0\cdot1288$ radian/sec.$^2=7\cdot73$ radians per second

$=7\cdot73\times60/2\pi=74$ revolutions per minute

(2) Alternatively considering the equality of the angular momentum gained and the angular impulse, if ω is the angular velocity after 60 sec.

$$\omega \times 1553 \text{ lb.-ft.-sec.}^2 = 200 \text{ lb.-ft.} \times 60 \text{ sec.}$$

$$\omega = \frac{200 \times 60}{1553} \cdot \frac{\text{lb.-ft.-sec.}}{\text{lb.-ft.-sec.}^2} = 7{\cdot}73 \text{ radians/sec.}$$

EXAMPLE 5. A body of weight W is drawn up a rough inclined plane by a force P inclined θ to the plane. Find the value of P to give the body an acceleration f up the track, the coefficient of friction between the body and the track being μ and the inclination of the plane being α.

Let $\tan \phi = \mu$, so that ϕ is the angle of friction. A graphical solution is shown on the right-hand side of Fig. 70.

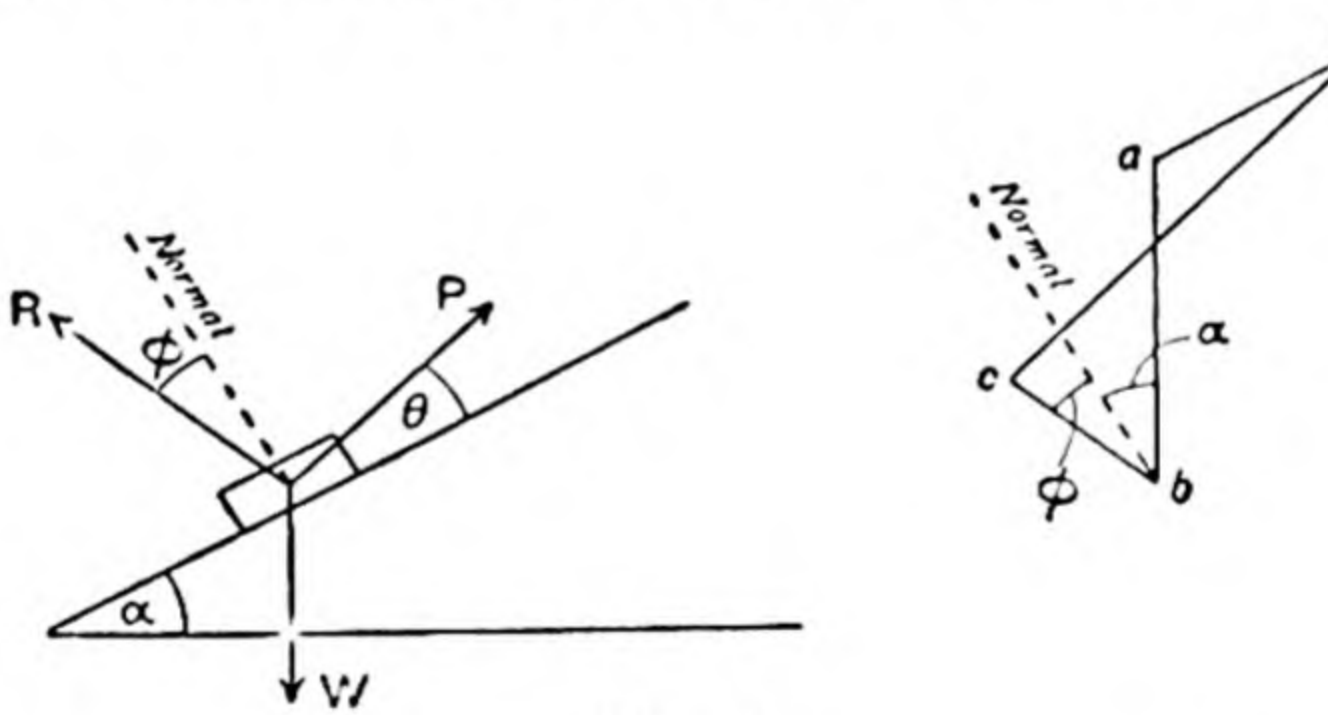

FIG. 70.

A vertical vector ab is drawn to represent W to scale and an indefinite line bc is drawn inclined ϕ to the normal to the plane or at $\alpha + \phi$ to ab. The thrust of the plane on the body is in the direction R (as in Fig. 39, Art. 40), but the length bc to represent R is not yet known. A vector cd is required, parallel to P and such as to give, with W and R, a resultant Wf/g, the unbalanced force required to give the body an acceleration f up the plane. This known resultant may be set off from a inclined α to the horizontal and of length to represent Wf/g: it is shown by the vector, ad. Then a line cd through d and parallel to P

meeting *bc* in *c* will give the vector representing P. In vectors, $ab+bc+cd=ad$. It is evident that the angle θ to give a minimum value of P is such that *cd* is perpendicular to *cb* (as in Art. 40, where $f=0$).

A trigonometrical expression for the value of P in terms of W can be deduced from the vector force diagram *abcd*, but we may readily find it by resolving the forces parallel to, and perpendicular to, the plane. Thus if we take the normal component of R to be N and the tangential component (down the plane) to be μN, resolving along the slope (upward) and equating this unbalanced force to the mass multiplied by the acceleration f,

$$P\cos\theta - W\sin\alpha - \mu N = Wf/g$$

And perpendicular to the slope, the acceleration being zero,

$$N - W\cos\alpha + P\sin\theta = 0, \text{ or } N = W\cos\alpha - P\sin\theta$$

Substituting this value of N in the previous equation

$$P\cos\theta - W\sin\alpha - \mu(W\cos\alpha - P\sin\theta) = Wf/g$$

$$P = W(f/g + \sin\alpha + \mu\cos\alpha)/(\cos\theta + \mu\sin\theta)$$

which may also be written

$$P = W\{f/g\cos\phi + \sin(\alpha+\phi)\}/\cos(\theta-\phi)$$

which reduces to the form (3), Art. 40, if $f=0$.

The value of P for a force parallel to the slope is found by writing $\theta=0$, and for a horizontal force by writing $\theta=-\alpha$.

If, in accordance with d'Alembert's Principle, we were to include a fictitious force Wf/g *down* the slope among the forces exerted on the body, the external forces would then be in equilibrium; and in the first equation, instead of Wf/g on the right-hand side, we should have $-Wf/g$ on the left-hand side and zero on the right. The mathematics, and the value of P obtained, would be unaltered. Also vectorially we should then have $ab+bc+cd+da=0$. But there is no point in so making the problem one in statics by an artifice which does not represent the real state of affairs.

EXAMPLE 6. A four-wheeled vehicle, Fig. 71, is driven by the rear wheels. The c.g. is at a height h above the track and distant a_1 horizontally from the front axle and a_2 from the rear axle. Find the greatest possible acceleration if the coefficient of friction between the wheels and track is μ, and find the load carried on the rear and on the front pairs of wheels during such acceleration.

Let R_1 and R_2 be the vertical forces exerted by the track on the front and rear wheels respectively, during acceleration. The driving force exerted by the track on

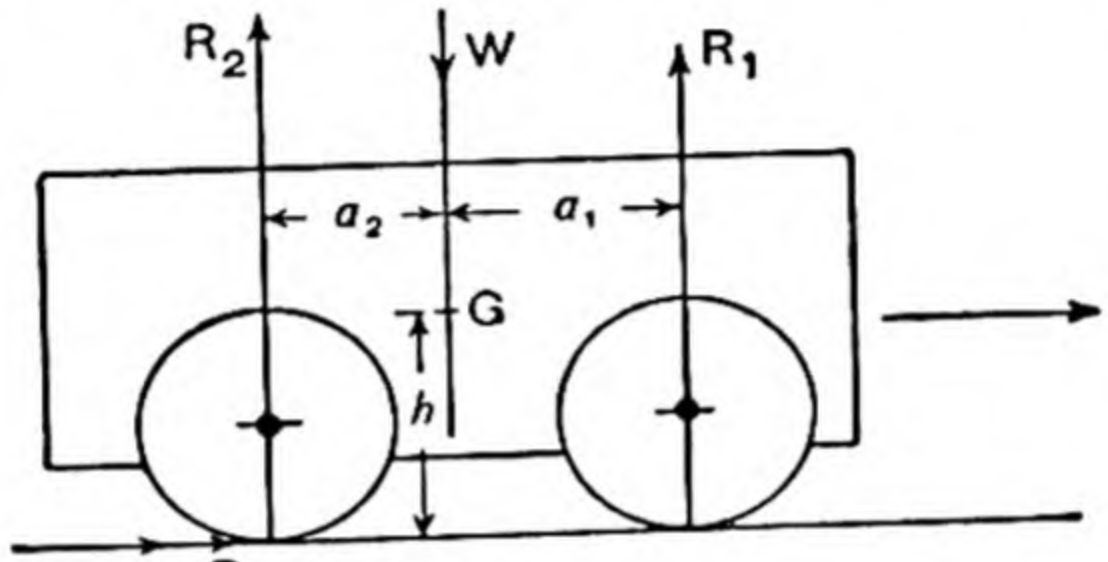

FIG. 71.

the rear wheels is then μR_2, and the external forces are as shown in Fig. 71. Let f be the acceleration in the direction of motion of the vehicle. Then from the second law of motion,

$$\mu R_2 = Wf/g$$

And in a vertical direction, the acceleration being zero,

$$R_1 + R_2 = W$$

And taking moments about an axis through G (the c.g.), and perpendicular to the diagram, there being no rotation or angular acceleration about this axis, the moments are equal and opposite

$$\mu R_2 h + R_1 a_1 = R_2 a_2$$

(Note that the moments of the external forces about a parallel axis are not generally zero. The external forces have a horizontal resultant Wf/g through G, the moment of

which is eliminated by choosing an axis through G, but not if a higher or a lower position were chosen. Here the problem differs from that of the truck at rest or at constant speed. If a fictitious horizontal backward force Wf/g through G were introduced, as justified by d'Alembert's Principle, we should then proceed as in statics, *i.e.* the horizontal and the vertical components would each be equated to zero and the moments of the external forces about any axis perpendicular to the diagram would be equated to zero.)

From the three equations above it is easy to find f, R_1, and R_2. Substituting for R_1 its value, $W-R_2$ (from the second equation), in the third equation,

$$\mu R_2 h+(W_1-R_2)a_1=R_2 a_2$$

$$R_2=W\frac{a_1}{a_1+a_2-\mu h}$$

This may also be written

$$R_2=Wa_1/(a_1+a_2)+\mu R_2 h/(a_1+a_2)$$

of which $Wa_1/(a_1+a_2)$ is the load on the back axle when the vehicle is stationary or moving at constant speed (or $\mu=0$), and the second term $\mu R_2 h/(a_1+a_2)$ is the load transferred to the back axle, from the front one, by the driving effort. The load R_1 on the front axle is therefore $Wa_2/(a_1+a_2)$ $-\mu R_2 h/(a_1+a_2)$, which is $W(a_2-\mu h)/(a_1+a_2-\mu h)$. The two equal and opposite forces $\mu R_2 h/(a_1+a_2)$ imposed downward on the back axle and upward on the front one constitute a clockwise couple $\mu R_2 h$ (with an arm a_1+a_2), balancing the counter-clockwise moment $\mu R_2 h$ of the driving force about the c.g.

The maximum acceleration, from the first equation, is

$$f=g\mu R_2/W=\mu g a_1/(a_1+a_2-\mu h)$$

If the back wheel were locked, μR_2 would be reversed in direction and would be the decelerating force. In this case,

$$R_2=Wa_1/(a_1+a_2+\mu h)$$

and the maximum deceleration is

$$f=\mu g a_1/(a_1+a_2+\mu h)$$

which is obviously smaller than the maximum acceleration above. Acceleration increases R_2 (and μR_2), while deceleration diminishes both R_2 and the maximum retarding force μR_2.

For any driving torque or braking torque short of the maximum at which slipping occurs, dividing the torque by the wheel radius we get a propelling or a retarding force F, say, less than μR_2, exerted by the track on the wheel; and the transfer of load from one axle to the other during acceleration or deceleration is $Fh/(a_1+a_2)$, while F is equal to Wf/g where f is the acceleration, so that the load transferred from one axle to the other is $Wh(f/g)/(a_1+a_2)$.

EXAMPLE 7. If the vehicle in Example 6 is drawn by a horizontal drawbar pull P at a height h' above the track, find the acceleration and retardation which would be sufficient to make the front and back wheels respectively leave the track (*i.e.* jump upwards).

$$\text{Horizontally, } P = Wf/g$$

$$\text{Vertically, } R_1 + R_2 = W$$

Taking moments about an axis perpendicular to Fig. 71 and through G, since P is at a distance $h-h'$ below G and since there is no rotation or angular acceleration about this axis,

$$P(h-h') + R_1 a_1 = R_2 a_2$$

From which equations R_1 and R_2 can be derived:

$$R_1 = Wa_2/(a_1+a_2) - P(h-h')/(a_1+a_2)$$

$$R_2 = Wa_1/(a_1+a_2) + P(h-h')/(a_1+a_2)$$

If h is greater than h', R_1 becomes zero when P reaches the value $Wa_2/(h_1-h')$ and f is then $g \cdot a_2/(h-h')$, and at this acceleration the front wheels are on the point of leaving the track; while if h' were greater than h, R_2 would reach the track; while if h' were greater than h, R_2 would reach zero and the back wheels would lift when P reaches $Wa_1/(h'-h)$ or f reaches $ga_1/(h'-h)$.

For retardation or deceleration, due to a backward thrust P, if h is greater than h', the back wheels would lift at a value $f = ga_2/(h'-h)$, and if h' is greater than h, the front wheels would lift at a value $f = ga_2/(h'-h)$.

By thinking of high values of h and then of h' the

general nature of what would happen with high acceleration or deceleration will be evident without calculation.

59. Third Law. (*a*) *Linear Motion.* To every force there is an equal and opposite force, often called a reaction. If a body A exerts a force P on a body B, this body B exerts an equal and opposite force P on A. Thus forces exist only in pairs, one force being exerted on a body and the other by it on some other matter. In considering the equilibrium or the acceleration of a body, we are generally concerned with the forces exerted on it; but in considering an assemblage of bodies in contact we may determine a force exerted on one from a knowledge of a force which it exerts on another. This law is true of bodies at rest or in motion.

(*b*) *Angular Motion.* To every torque there is an equal and opposite torque, often called a torque reaction. If a body A exerts a torque T on a body B, then B exerts an equal and opposite torque T on A. Thus torques also exist only in pairs.

60. Conservation of Momentum. An important consequence of the third law is that the total momentum of the two bodies is unaltered by any mutual action between them. For since the force exerted by A on B is the same as that exerted by B on A, the impulse during any interval given by A to B is of the same amount as that given by B to A and in the opposite direction. Hence, if B gains any momentum, A loses exactly the same amount, and the total change of momentum is zero, and this is true for any and every direction. This is expressed by the statement that for any isolated system of bodies momentum is constant. Thus when a projectile is fired from a gun, the impulse or change of momentum of the shot due to the explosion is of equal amount to that of the recoiling gun in the opposite direction. The momentum of the recoil is transmitted to the earth, and so is that of the shot, the net momentum given to the earth being also zero.

The conservation of momentum is unaffected by friction. A frictional drag acting on a body against its direction of motion is a force by which the momentum is given from the body to the earth just as by a direct thrust or pull.

EXAMPLE. A gun weighing 30 tons fires a 1000-lb. projectile with a velocity of 1000 feet per second. With what initial velocity will the gun recoil? If the recoil is overcome by a (time) average force of 60 tons, how far will the gun travel? How long will it take?

Let v be the initial velocity of the gun in ft. per sec. Using units of lb. mass and ft./sec. velocity and equating the momentum of the projectile to that of the gun in the opposite direction,

$$1000 \text{ lb.} \times 1000 \text{ ft./sec.} = 30 \times 2240 \text{ lb.} \times v$$
$$v = 1{,}000{,}000/6720 = 14{\cdot}88 \text{ ft./sec.}$$

Then if t = time of recoil, adopting lb. units of force and, lb.(force)/g, units of mass

$$\text{Impulse of retarding force} = 60 \times 2240 \text{ lb.} \times t$$

$$\text{Momentum of gun (or of shot)} = 1000 \times 1000/32{\cdot}2 \ \frac{\text{lb.-ft.} \times \text{sec.}^2}{\text{sec.} \times \text{ft.}}$$
$$= 1{,}000{,}000/32{\cdot}2 \text{ lb.-sec.}$$

And equating the impulse to this,

$$60 \times 2240 \text{ lb.} \times t = 1{,}000{,}000/32{\cdot}2 \text{ lb.} \times \text{sec.}$$
$$t = 0{\cdot}231 \text{ second.}$$
$$\text{Travel of gun} = \tfrac{1}{2}vt = \tfrac{1}{2} \times 14{\cdot}88 \times 0{\cdot}231 = 1{\cdot}72 \text{ ft.}$$

61. Motion of Connected Weights. In various lifts or hoists or in winding gear, heavy masses move up and down with considerable accelerations controlled by ropes or cables to which they are attached. In such cases the tensions in the ropes are not necessarily equal to the weights of the attached masses as they are in ropes carrying suspended masses at rest. Even very simple cases of masses in vertical motion under the control of cords or ropes illustrate the second and third laws of motion and the principles applicable to less simple cases. Suppose two weights W_1 lb. and W_2 lb. to be connected by a light inextensible cord passing over a perfectly smooth pulley as in Fig. 72. Such an arrangement, on an experimental

scale, with various devices for measuring time and displacements constitutes an Atwood's machine used to illustrate the laws of motion under gravity. (But the smooth pulley is replaced by a small light one on ball bearings.)

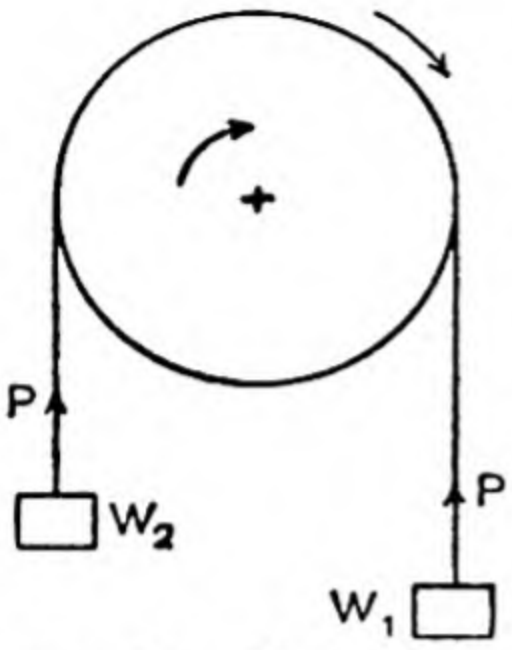

FIG. 72.—Motion of connected weights.

If W_1 is greater than W_2, W_1 will move downward and W_2 upward, with a common acceleration f, say. Let P be the tension in the cord.

Consider W_1: the downward force on it is W_1 (its weight), and the upward force is P, which is the same throughout the string; hence the downward accelerating force is $W_1 - P$.

Hence (by Art. 58),

$$W_1 f/g = W_1 - P \quad . \quad . \quad (1)$$

Similarly, on W_2 the *upward* accelerating force is $T - W_2$;

$$\text{hence } W_2 f/g = P - W_2 \quad . \quad . \quad . \quad . \quad (2)$$

adding (1) and (2)

$$(W_1 + W_2) f/g = W_1 - W_2 \quad . \quad . \quad . \quad (3)$$

$$\text{or} \quad f = \frac{W_1 - W_2}{W_1 + W_2} \cdot g \quad . \quad . \quad . \quad (4)$$

and from (1)

$$P = W_1(1 - f/g) = 2W_1W_2/(W_1 + W_2) \quad . \quad (5)$$

The acceleration f might have been stated from considering the two weights and string as one complete system. The external accelerating force on the system is $W_1 - W_2$, and its mass is $(W_1 + W_2)/g$;

$$\text{hence } f = \frac{\text{accelerating force}}{\text{total mass}} = \frac{W_1 - W_2}{W_1 + W_2} \cdot g \quad . \quad (6)$$

Note that in deducing (4) we eliminated P from (1) and (2) as a matter of mathematics. But in obtaining f at (6), we eliminated P by making it an internal force which has no effect on the momentum of the system of two masses.

As a further example, suppose a body of weight W_2 is

drawn along a horizontal track by a weight W_1 descending vertically and attached to W_2 by a cord passing over a small pulley of negligible mass and negligible friction as shown in Fig. 73. Let μ be the coefficient of friction between W_2 and the track, so that the resistance to motion is

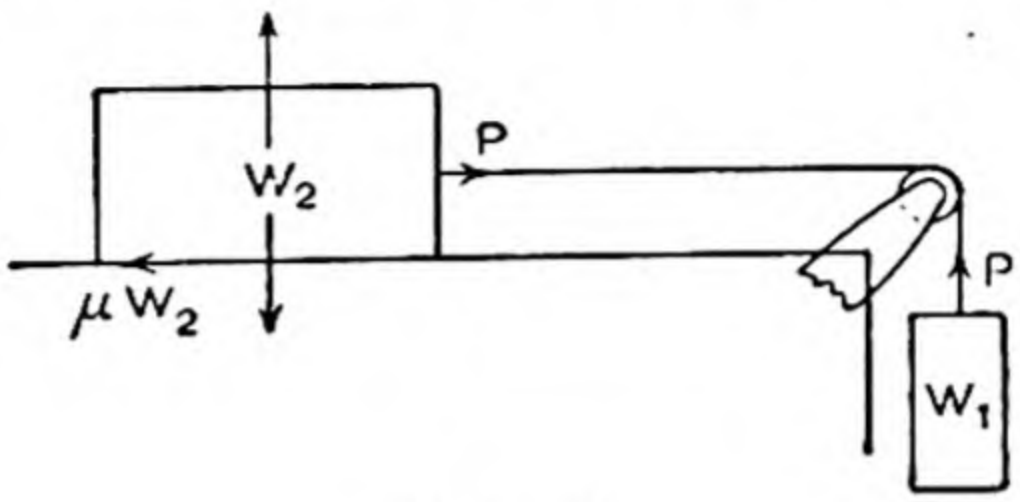

FIG. 73.

μW_2. Then considering in turn the common acceleration f in relation to W_1 and W_2,

$$f/g=(P-\mu W_2)/W_2=(W_1-P)/W_1 \quad . \quad (7)$$

$$P(W_1+W_2)=W_1W_2+\mu W_1W_2=W_1W_2(1+\mu) \quad . \quad (8)$$

$$P=W_1W_2(1+\mu)/(W_1+W_2) \quad . \quad . \quad . \quad (9)$$

and

$$f=\frac{W_1-\mu W_2}{W_1+W_2}\cdot g \quad . \quad . \quad . \quad . \quad . \quad (10)$$

results which have obvious simplifications if $\mu=0$, *i.e.* if the track is perfectly smooth.

Here again we could simplify the work of finding f by considering the two masses as one system in which P is an internal force, and which being dual makes no contribution to any change of momentum. The external force in the direction of motion is $W_1-\mu W_2$, and on the mass W_1+W_2, the acceleration is evidently

$$f=\frac{W_1-\mu W_2}{W_1+W_2}\cdot g$$

62. Combined Linear and Rotary Inertia. In winding and lift problems we are often concerned with ascending and descending masses, which have linear accelerations, in conjunction with flywheels which have rotary motion and

acceleration simply related to the linear accelerations of the ascending and descending masses. The principles can be illustrated by simple examples in which the only motive force is a descending weight. We can express the linear velocity and acceleration of the masses having rectilinear motion in terms of the angular velocity and acceleration of the axle and wheels, or we can express the angular velocity and acceleration in terms of the linear motion. Another simple method of dealing with these problems by consideration of energy is given in Art. 66.

EXAMPLE 1. *Flywheel, Axle, and Driving Weight.* A flywheel, together with the axle (of radius r) to which it is

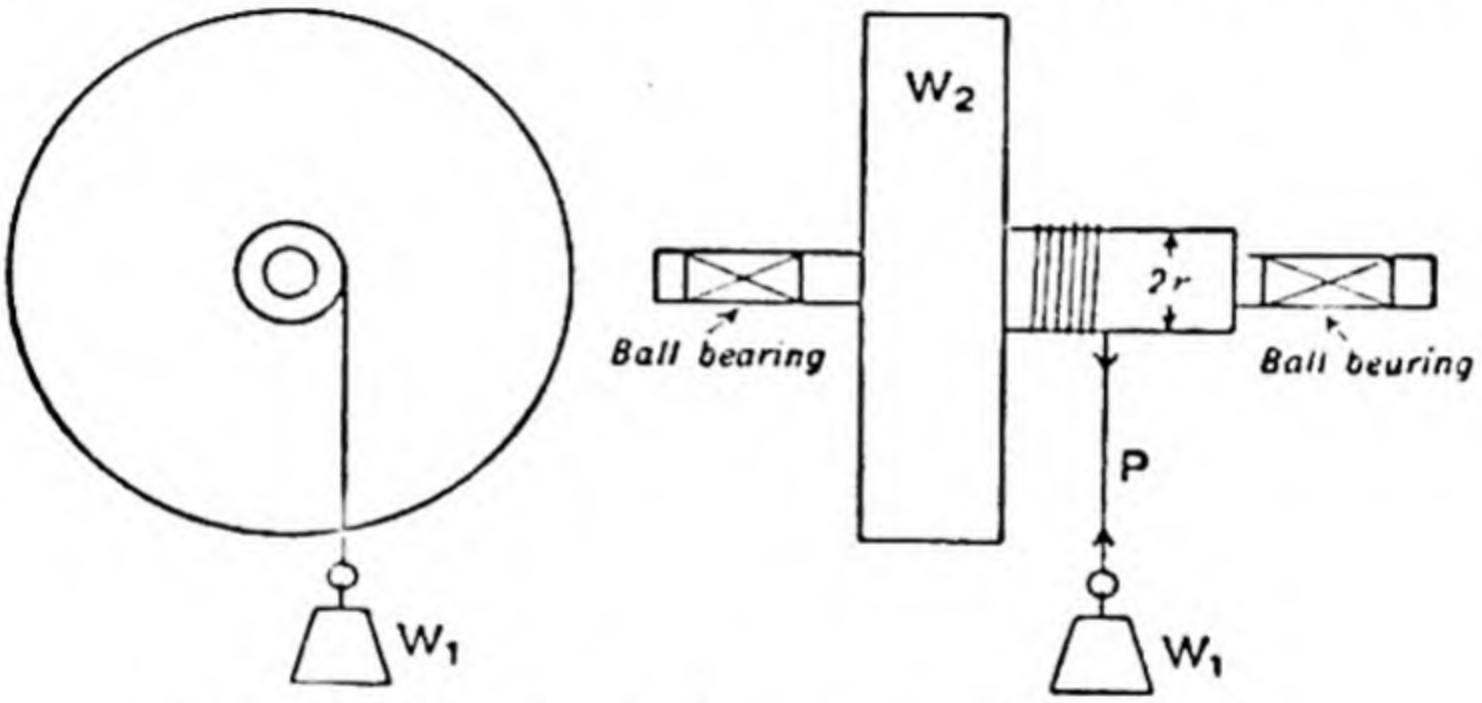

FIG. 74.—Combined linear and rotary inertia.

attached, weighs W_2 and is accelerated by a descending weight W_1 attached to a cord wrapped round the axle as shown in Fig. 74. Find the acceleration of the weight and wheel and the tension (P) of the cord.

Let α be the angular acceleration of the wheel. Then $\alpha r = f$, say, is the acceleration of the weight W_1. Let k be the radius of gyration of the combined wheel and axle. The effective downward force on W_1 is $W_1 - P$, hence by the second law of motion ((1), Art. 58)

$$W_1 f/g = W_1 - P \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The driving torque T on the rotary parts is, from (3), Art. 58,

$$T = Pr = \text{mass} \times k^2 \times \alpha, \text{ or, } (W_2 k^2/g)(f/r) \quad . \quad (2)$$

Equating the two values of P from (1) and (2),

$$P = W_1 - W_1 f/g = W_2 k_2 f/(gr^2) \quad . \quad . \quad (3)$$

$$f = \frac{W_1}{W_2(k^2/r^2) + W_1} \cdot g = g \Big/ \left(\frac{W_2\ k^2}{W_1\ r^2} + 1 \right) \quad . \quad (4)$$

and

$$a = f/r = (g/r) \Big/ \left(\frac{W_2\ k^2}{W_1\ r^2} + 1 \right) \quad . \quad . \quad . \quad (5)$$

And substituting for f in (2) or (1),

$$P = \frac{W_2(k^2/r^2)}{1 + W_2 k^2/(W_1 r^2)} \text{ or } W_1 \frac{W_2 k^2}{W_1 r^2 + W_2 k^2} \quad . \quad (6)$$

Note that in the *ratios* W_2/W_1, k^2/r^2, the units are of no importance; so long as they are consistent, then f will be in the same units as g, and P will be in the same units as W_1 and W_2.

Here we considered separately the motions of W_1 and W_2, P being an external force to each. But dynamically, W_1, at a radius r, might be considered as equivalent to an addition to the rotary inertia, the whole system having exerted upon it the external torque $W_1 r$. Then from (3),

Art. 58

$$a(W_2 k^2 + W_1 r_1^2)/g = W_1 r \quad . \quad . \quad . \quad (7)$$

$$a = g \,.\, W_1 r/(W_2 k^2 + W_1 r^2) \quad . \quad . \quad . \quad (8)$$

which is the same as (5), and $f = ar$ is the same as (4). This solution is shorter, though perhaps not so obvious, and does not give P, though it may be found from (1) when f has been determined.

EXAMPLE 2. *Flywheel, Axle, and Two Weights.* A flywheel which, together with an axle, has a weight W_3 and radius of gyration k, is driven by a descending weight W_1 attached to a cord wrapped round the axle at a radius r_1. A second weight W_2 attached to a cord wrapped round the axle at a radius r_2 is lifted when W_1 descends. Find the accelerations and the tensions in the two cords (see Fig. 75). Also indicate the nature of the modifications if an external torque T′ is exerted on the axle.

Let P_1 be the tension of the cord attached to W_1

and P_2 " " " " W_2

Considering the angular acceleration (a) of the rotary parts

$$\text{Torque}=T=P_1r_1-P_2r_2=W_3k^2a/g \quad . \quad . \quad (9)$$

And for the accelerated motion of W_1 and W_2, let f_1 be the downward acceleration of W_1 and f_2 be the upward acceleration of W_2

Then, $$P_1=W_1-W_1f_1/g=W_1-W_1ar_1/g \quad . \quad . \quad (10)$$

and $$P_2=W_2+W_2f_2/g=W_2+W_2ar_2/g \quad (11)$$

Hence,

$$P_1r_1=W_1r_1(1-ar_1/g)$$
$$P_2r_2=W_2r_2(1+ar_2/g)$$

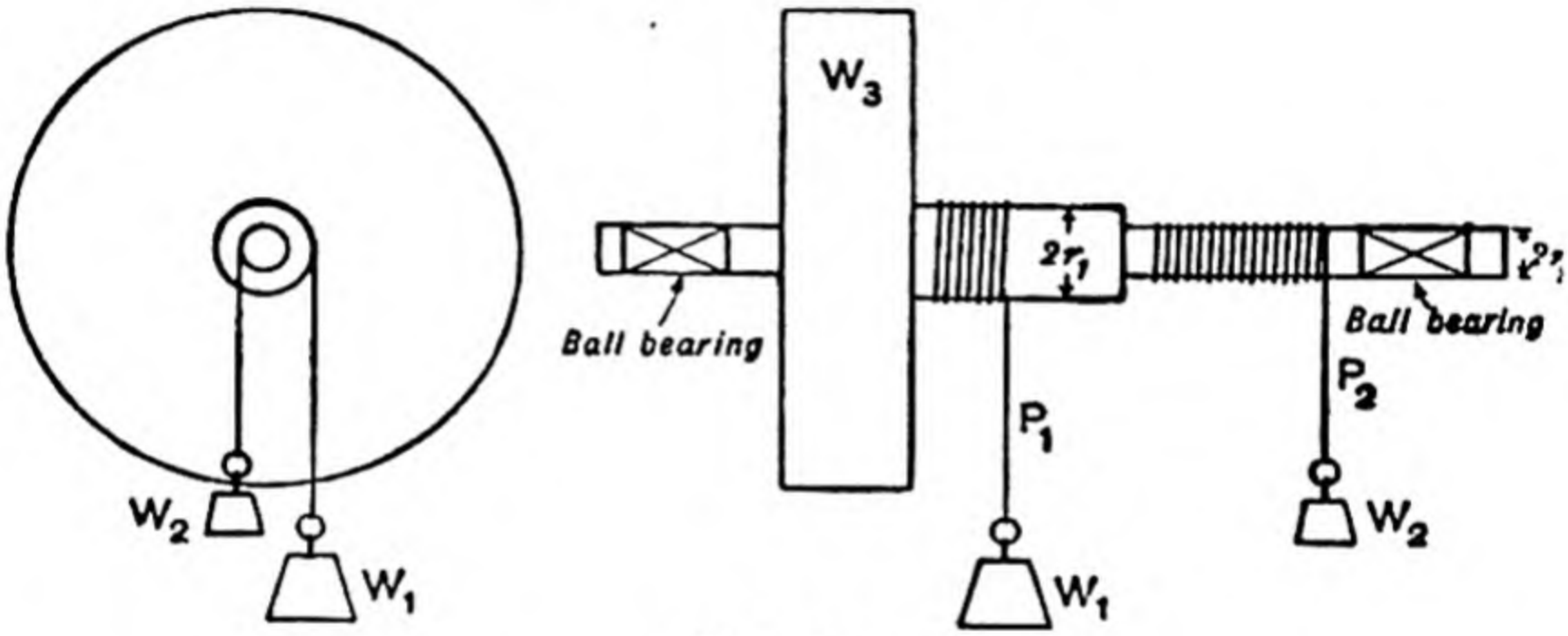

FIG. 75.

by subtraction,

$$P_1r_1-P_2r_2=W_1r_1-W_2r_2-(W_1r_1^2+W_2r_2^2)a/g \; . \quad (12)$$

which from (9) is equal to W_3k^2a/g, hence

$$a(W_3k^2+W_1r_1^2+W_2r_2^2)/g=W_1r_1-W_2r_2 \; . \quad (13)$$

$$a=\frac{(W_1r_1-W_2r_2)}{W_3k^2+W_1r_1^2+W_2r_2^2}\cdot g \quad . \quad . \quad . \quad (14)$$

This result is also evident from considering the whole system W_1, W_2, and W_3; the moment of the external forces is $W_1r_1-W_2r_2$ and the effective inertia is $1/g$ times $W_3k^2+W_1r_1^2+W_2r_2^2$, for W_1 and W_2 have the same dynamic effect as if they were concentrated at radii r_1

and r_2 respectively. A mistake to be guarded against is that of supposing that the torque on the rotary system is $W_1r_1-W_2r_2$. It is not. The torque on the purely rotary parts is $P_1r_1-P_2r_2$, and in this, P_1 is less than W_1, while P_2 is greater than W_2 (if W_1r_1 is greater than W_2r_2, so that the angular acceleration is such that W_1 is accelerated downwards and W_2 upwards).

The accelerations $f_1=ar_1$ and $f_2=ar_2$ easily follow. And from (10) and (11)

$$P_1=W_1(1-ar_1/g) \quad . \quad . \quad . \quad . \quad (15)$$

$$P_2=W_2(1+ar_2/g) \quad . \quad . \quad . \quad . \quad (16)$$

If a constant torque T′ is applied to the axle by some form of motor, equation (9) will have T′ added to $P_1r_1-P_2r_2$, and a, f_1 and f_2 will be increased. The right-hand side of equation (13) will be increased by T′. Evidently, then, from (10) and (11), P_1 will be reduced and P_2 increased. If T′ is negative (as with a frictional or braking torque) a will be decreased and will be negative if T′ is greater than $W_1r_1-W_2r_2$ and hence P_1 will be increased and P_2 reduced. The angular acceleration will be

$$a=g(T'+W_1r_1-W_2r_2)/(W_3k^2+W_1r_1^2+W_2r_2^2) . \quad (17)$$

and the winding tensions P_1 and P_2 can then be found from (10) and (11) as before, as shown at (15) and (16).

The accelerations and tensions for a wheel and axle when the rotary inertia of the wheel and axle is negligible can evidently be found by putting W_3 equal to zero in the foregoing results.

If, say, W_1, instead of winding off a larger diameter of axle or drum than W_2, is connected with the axle of W_2 by any form of gearing so that W_1 runs n times as fast as W_2, provided the rotary inertia of the gearing is negligible, the preceding results for a, f_1, and f_2 will hold good if we write nr_2 in place of r_1 and $n^2r_2^2$ in place of r_1^2.

And if instead of the drive being from the descent of the weight W_1, a winding torque T′ were applied, T′ would replace W_1r_1, and $W_1r_1^2$ would also be zero. Then we should have

$$a=g(T'-W_2r_2)/(W_3k^2+W_2r_2^2) \quad . \quad . \quad (18)$$

and the winding tension P_2 would be as given in (11) and (16) with the value of a from (18).

EXAMPLE 3. A truck when loaded weighs 2500 lb. and is hauled up an incline of 1 in 25 by a rope parallel to the incline. The rope is wrapped round a winding drum 4 feet diameter mounted on a shaft which also carries a flywheel weighing in all 3000 lb. and having a radius of gyration of 1·8 ft. Neglecting the weight of the winding ropes, rotary inertia of the truck wheels and friction, find the torque necessary on the shaft to give the truck an acceleration of 0·8 ft./sec.^{2} up the incline and find the tension in the rope.

Let P be the tension. The gravitational pull on it will be 1/25 of 2500 lb. $=100$ lb. Then for an acceleration of 0·8 ft./sec.^{2}, the effective pull will be

$$P - 100 \text{ lb.} = 0{\cdot}8 \times 2500/32{\cdot}2 \text{ lb.}$$

$$P = 100 + 62{\cdot}1 = 162{\cdot}1 \text{ lb.}$$

The angular acceleration of the drum, shaft, and flywheel will be (0·8 ft./sec.^{2})/2 ft. radius $=0{\cdot}4$ radians/sec.^{2} And the torque T necessary to give this angular acceleration and the pull P will be

$$T = 162{\cdot}1 \text{ lb.} \times 2 \text{ ft.} + \frac{3000 \text{ lb.}}{32{\cdot}2 \text{ ft./sec.}^2} \times \frac{1{\cdot}8^2 \text{ft.}^2}{1} \times \frac{0{\cdot}4 \text{ radian}}{\text{sec.}^2}$$

$$= 324{\cdot}2 + 3888/32{\cdot}2 \text{ lb.-ft.}$$

$$= 324{\cdot}2 + 120{\cdot}7 = 445 \text{ lb.-ft.}$$

63. Conservation of Energy. The principle that energy cannot be destroyed, though it may be transformed into a different kind of energy, is based on experience and serves to solve many problems. In mechanics we are mainly concerned with (1) potential energy, such as that possessed by a body raised above a datum level to which it can subsequently descend, doing work, or an elastically strained spring which can do work in relaxing to its former condition, and (2) kinetic energy or energy of motion—often written in the abbreviated form K.E. Mechanical energy is transformed into heat energy when frictional resistance is overcome; it may be transformed into electrical energy by driving an electrical generator; we know the rates of exchange into these other forms of energy, but such transformed energy does not further concern us in applied mechanics.

We have had an example of the conservation of energy

in the energy expended and the energy usefully employed in a simple frictionless machine in Art. 46. In this an important condition, in addition to the absence of friction, was that there was no acceleration and therefore no change in kinetic energy, *i.e.* no energy exerted was converted into kinetic energy.

64. Kinetic Energy. (*a*) *Linear Motion.* If a force P is exerted in accelerating a weight W, the acceleration f produced is from Art. 58, $f=P/m$ or Pg/W. If the motion is in the direction of P and if v_1 is the initial speed, the final speed v_2 after, say, a time t is $v_2=v_1+ft$, and the distance moved is

$$s=\tfrac{1}{2}(v_2+v_1)t \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$s=\tfrac{1}{2}(v_2+v_1)\times(v_2-v_1)/f$$

or
$$s=\tfrac{1}{2}(v_2^2-v_1^2)/f \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or
$$2fs=v_2^2-v_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The work done and energy expended in the exertion of the force P through a distance s is

$$P\,.\,s=P\times\tfrac{1}{2}(v_2^2-v_1^2)/f=\tfrac{1}{2}W(v_2^2-v_1^2)/g \quad . \quad (4)$$

which is the increase of K.E. And if $v_1=0$, then the kinetic energy is

$$K.E.=\tfrac{1}{2}Wv_2^2/g \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(*b*) *Rotation.* If a body has an angular acceleration α about an axis, due to a torque T, and its moment of inertia about the axis is I or Wk^2/g, then $\alpha=T/I$ or $Tg/(Wk^2)$. If ω_1 is the initial and ω_2 the final angular velocity after an interval of time t, $\omega_2=\omega_1+\alpha t$ and the angular displacement or movement θ is

$$\theta=\tfrac{1}{2}(\omega_2+\omega_1)t \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\theta=\tfrac{1}{2}(\omega_2+\omega_1)(\omega_2-\omega_1)/\alpha$$

$$\theta=\tfrac{1}{2}(\omega_2^2-\omega_1^2)/\alpha \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or
$$2\alpha\theta=\omega_2^2-\omega_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The work done and energy expended in the exertion of the torque T through the angle θ is

$$T\theta=T\times\tfrac{1}{2}(\omega_2^2-\omega_1^2)/\alpha=\tfrac{1}{2}I(\omega_2^2-\omega_1^2)$$

or
$$=\tfrac{1}{2}Wk^2(\omega_2^2-\omega_1^2)/g \quad . \quad . \quad . \quad . \quad . \quad (9)$$

which is the increase in K.E. And if $\omega_1 = 0$,

$$\text{K.E.} = \tfrac{1}{2}\text{I}\omega_2^2 \text{ or } \tfrac{1}{2}\text{W}k^2\omega_2^2/g \quad . \quad . \quad . \quad (10)$$

The analogy between the relations for linear and angular motion is complete.

In (4) and (9) the work done by a constant force and a constant torque respectively is in each case equal to the gain of K.E. But these relations remain true if the force P or the torque T are variable quantities, provided P and T denote the mean values of the force or torque respectively, the mean being not with respect to time but with respect to displacement, *i.e.* to distance and angle respectively. In other words, for a small displacement δs, the work done is $\text{P} \,.\, \delta s$ and the gain in K.E. is $(\tfrac{1}{2}\text{W}/g) \times \delta(v^2)$, and over a larger displacement

$$\Sigma(\text{P}\delta s) = \Sigma\{(\tfrac{1}{2}\text{W}/g) \times \delta(v^2)\} = \tfrac{1}{2}(\text{W}/g)(v_2^2 - v_1^2) \quad (11)$$

where P is a variable quantity. And for a variable torque T, $\text{T} \,.\, \delta\theta = (\text{W}k^2/g) \times \delta(\omega^2)$ and

$$\Sigma(\text{T}\delta\theta) = \tfrac{1}{2}\,\Sigma\{(\text{W}k^2/g) \times \delta(\omega^2)\} = \tfrac{1}{2}(\text{W}k^2/g)(\omega_2^2 - \omega_1^2) \quad (12)$$

The relation (11) is shown graphically in Fig. 76, where the effective accelerating force P is shown as the excess of the driving forces such as PN above the resisting forces such as P'N, and the area MPP'M' represents the effective work done in accelerating a body and the gain in K.E. by the body. If the resistances exceed the driving effort, the area between the graphs of effort and resistance would have to be reckoned negative.

A weight oscillating to and fro, straining and relaxing a spring of any kind, is a good example of energy being repeatedly transformed from K.E. to some form of potential energy. In this case the strain energy temporarily stored in the spring and restored to the weight as K.E., reaches a maximum at mid-stroke and falls to zero at the ends of its stroke when all the energy is transferred to the spring (see Art. 74 and Fig. 94). And in a pendulum there is a similar repeated exchange between K.E. and the potential energy stored in the pendulum lifted against gravity, this potential energy reaching a maximum when the pendulum reaches its highest position and is instantaneously at rest with no K.E.

EXAMPLE 1. A train weighing 120,000 lb. moves at 60 miles per hour. Find its K.E. If it gets up speed from rest with a constant acceleration of 2 ft./sec.^{2}, what average power is required apart from that needed for frictional or gravitational resistances? Neglect the rotary inertia of the wheels. If the power is shut off and the train subject to a constant retarding force equal to 7·5 per

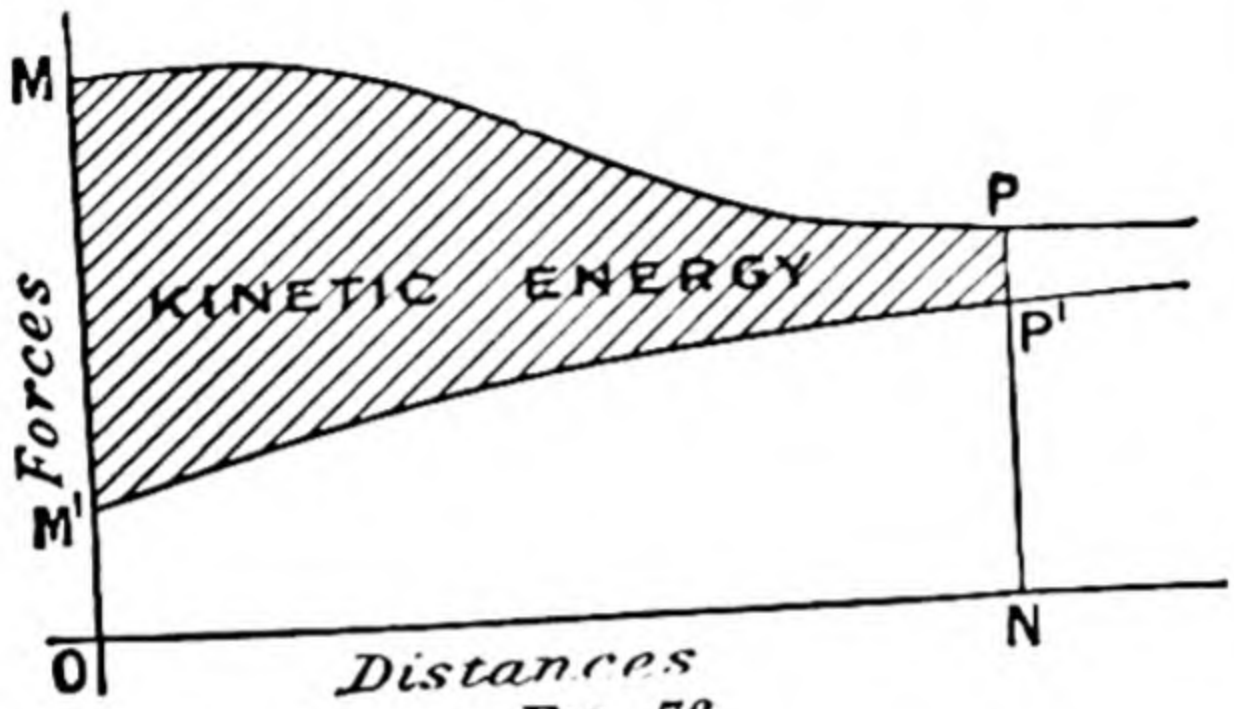

FIG. 76.

cent. of its weight, how far will it run before coming to rest?

Since the speed is 88 ft./sec.

$$\text{K.E.} = \tfrac{1}{2} \cdot 88^2 (ft./\text{sec.})^2 \times 120{,}000 \text{ lb.}/(32{\cdot}2 \text{ ft./sec.}^2)$$
$$= 14{,}430{,}000 \text{ ft.-lb.}$$

Time to get up speed $= 88$ (ft./sec.) $\div 2$ ft./sec.$^2 = 44$ sec.

Average work per sec. $= 14{,}430{,}000$ ft.-lb./44 sec.

Average $= 14{,}430{,}000$ ft.-lb./(44 sec. $\times 550$ ft.-lb./h.p.-sec.)

,, h.p. $= 596$ h.p.

If there were actually uniform acceleration, the maximum power required would be twice this average amount since the maximum speed would be twice the average speed. Actually accelerations are not constant, but this example serves to show that high power is necessary for the high accelerations in, say, an electric tube railway with high average speed and stops at short intervals; the energy is converted into heat in deceleration by braking. The

power greatly exceeds that necessary to overcome resistances of a frictional kind, *e.g.* if these were as high as one per cent. of the load, *i.e.* 1200 lb., the maximum h.p. required to overcome this would be

$$1200 \text{ lb.} \times 88 \text{ ft./sec.} \div 550 \text{ ft.-lb./h.p.-sec.} = 192 \text{ h.p.}$$

A resistance of 7·5 per cent. of 120,000 lb. = 9000 lb. This multiplied by the distance run is equal to the work done in absorbing the K.E. Hence,

$$\text{distance run} = 14{,}430{,}000 \text{ ft.-lb.}/9000 \text{ lb.} = 1603 \text{ ft.}$$

Alternatively from the second law, Art. 58, equation (1),

$$120{,}000 \text{ lb.} \times f/(32{\cdot}2 \text{ ft./sec.}^2) = 9000 \text{ lb.}$$

$$f = 2{\cdot}415 \text{ ft./sec.}^2$$

$$\text{time taken} = 88 \text{ ft./sec.} \div 2{\cdot}415 \text{ ft./sec.}^2 = 36{\cdot}44 \text{ sec.}$$

$$\text{distance} = \tfrac{1}{2} \times 88 \text{ ft./sec.} \times 36{\cdot}4 \text{ sec.} = 1603 \text{ ft.}$$

We may note that to find this distance the former method is more direct and shorter.

EXAMPLE 2. A rail truck weighing 3 tons moves at 10 ft. per second, meets and collides with a truck weighing 2 tons moving at 5 ft. per second in the opposite direction. If the two trucks then move together, find their common speed and how much K.E. has been lost.

The total momentum in the direction of the first truck is unaltered, hence if v is the speed in that direction after collision,

$$(3 \times 10) - (2 \times 5) = (3 + 2)v$$

$$v = 4 \text{ ft./sec.}$$

The K.E. before collision is

$$(3 \times 10^2 + 2 \times 5^2)/64{\cdot}4 \text{ ft.-tons} = 350/64{\cdot}4 \text{ ft.-tons}$$

$$\text{K.E. after collison} = 5 \times 4^2/64{\cdot}4 = 80/64{\cdot}4 \text{ ft.-tons}$$

$$\text{Loss of K.E.} = (350 - 80)/64{\cdot}4 = 270/64{\cdot}4 = 4{\cdot}193 \text{ ft.-tons}$$

$$= 9391 \text{ ft.-lb.}$$

EXAMPLE 3. A car weighs 12·88 tons, and starts from rest; the resistance of the rails may be taken as constant

and equal to 500 lb. After it has moved s feet from rest, the tractive force, P lb., exerted by the motors is as follows:—

s .	0	20	50	80	110	130	160	190	200
P .	1280	1270	1220	1110	905	800	720	670	660

Find the speed of the car after it has gone 200 feet from rest; also find the velocity at various intermediate points, and plot a curve of speed on a base of space described.

Plot the curve of P and s as in Fig. 77, and read off the

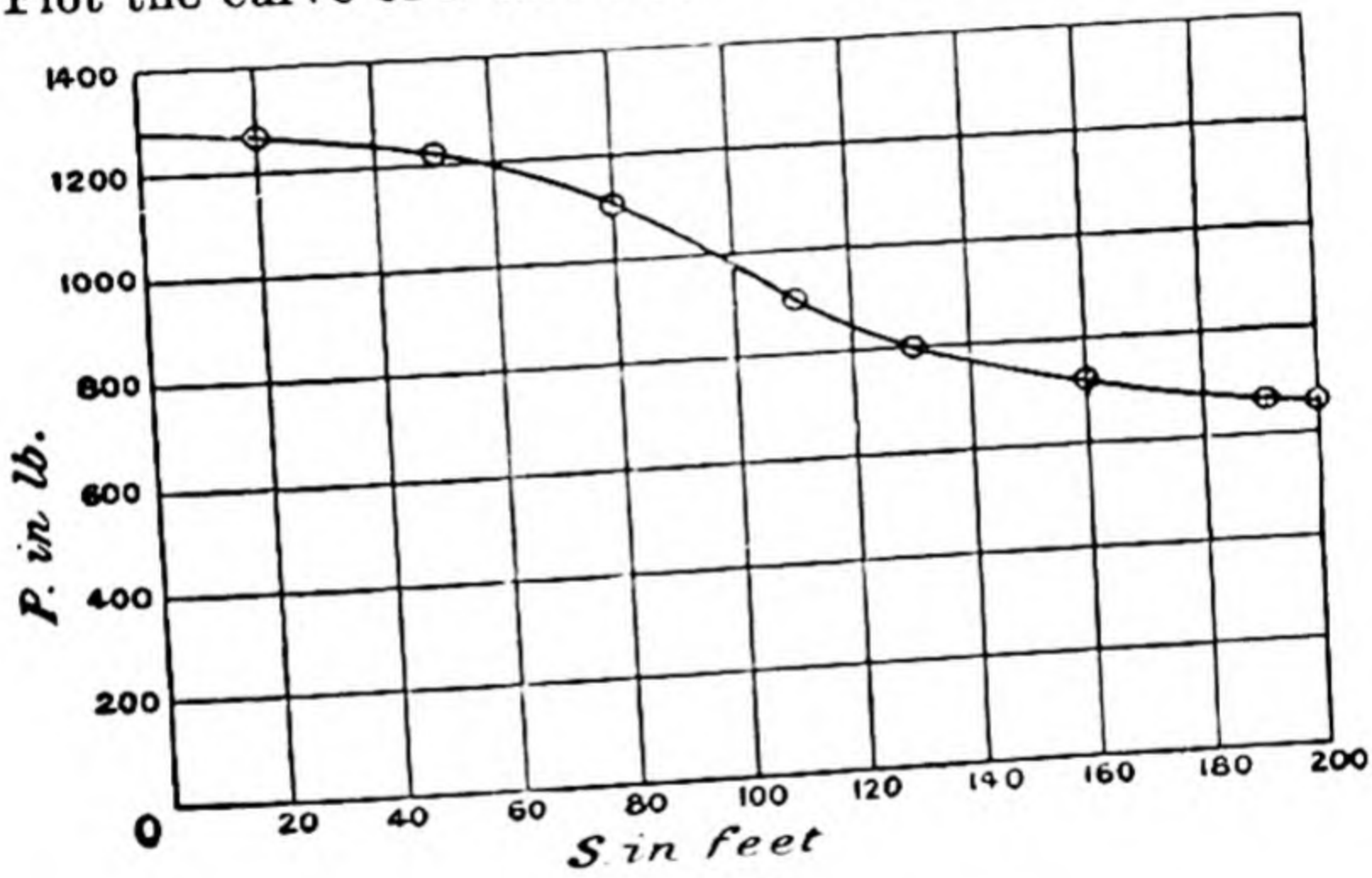

FIG. 77.

force every 20 feet, say, starting from $s=10$, and subtract 500 lb. resistance from each, as follows:—

s .	10	30	50	70	90	110	130	150	170	190
P .	1275	1260	1220	1150	1050	905	800	740	695	670
P − 500	775	760	720	650	550	405	300	240	195	170

The mean accelerating force during the first 20 feet of motion is approximately equal to that at $s=10$, viz. 775 lb.;

hence the work stored as kinetic energy (K.E.), *i.e.* the gross work done less that spent against resistance is

$$775 \text{ lb.} \times 20 \text{ ft.} = 15{,}500 \text{ ft.-lb.}$$

Then if v is the speed after covering s feet, for $s = 20$ ft.

$$\text{K.E.} = \tfrac{1}{2} W v^2/g = 15{,}500 \text{ ft.-lb.}$$

and W is $12{\cdot}88 \times 2240$ lb., hence

$$v^2 = 15{,}500 \text{ ft.-lb.} \times 64{\cdot}4 \text{ ft./sec.}^2 \div (12{\cdot}88 \times 2240) \text{ lb.}$$

$$= 34{\cdot}8 \text{ (ft./sec.)}^2$$

$$v = 5{\cdot}90 \text{ ft./sec.}$$

Similarly, the gain of K.E. in each 20 feet interval gives v^2 and v, *e.g.* in the next interval

$$\text{gain of K.E.} = 760 \text{ lb.} \times 20 \text{ ft.} = 15{,}200 \text{ ft.-lb.}$$

Total K.E. at $s = 40$ is $15{,}500 + 15{,}200 = 30{,}700$ ft.-lb. and so on, from which we can tabulate as follows :—

s in feet .	0	20	40	60	80	100	120	140	160	180	200
Gain of K.E. in 20 feet, foot-lb.	0	15500	15200	14400	13000	11000	8100	6000	4800	3900	3400
Total K.E., foot-lb.	0	15500	30700	45100	58100	69100	77200	83200	88000	91900	95300
v^2 . .	0	34·8	68·5	100·6	129·4	154·0	172·1	185·5	196·2	204·8	212·5
v in ft. per sec. .	0	5·90	8·28	10·03	11·34	12·40	13·12	13·62	14·01	14·30	14·58

These velocities have been plotted on a base of spaces in Fig. 78.

EXAMPLE 4. A wheel weighs 6000 lb. and has a radius of gyration of 3 ft. 6 in. Find its K.E. at 180 r.p.m. What uniform torque will be necessary to give it this speed, from rest, in 50 revolutions? What average h.p. will be required and what will be the maximum h.p.?

Let T be the torque in lb.-ft.

$$\text{Angular velocity} = 2\pi \times 180/60 = 6\pi \text{ radians/sec.}$$

$$\text{K.E.} = \tfrac{1}{2} W k^2 \omega^2/g$$

$$= 3000 \text{ lb.} \times 3{\cdot}5^2 \text{ ft.}^2 \times 36\pi^2 \text{ rad./sec.}^2 \div 32{\cdot}2 \text{ ft./sec.}^2$$

Equating the work done to the K.E.

$$T \times 50 \times 2\pi = 3000 \times 12{\cdot}25 \times 36\pi^2/32{\cdot}2 \text{ ft.-lb.}$$

$$= 405{,}500 \text{ ft.-lb.}$$

$$T = 1291 \text{ lb.-ft.}$$

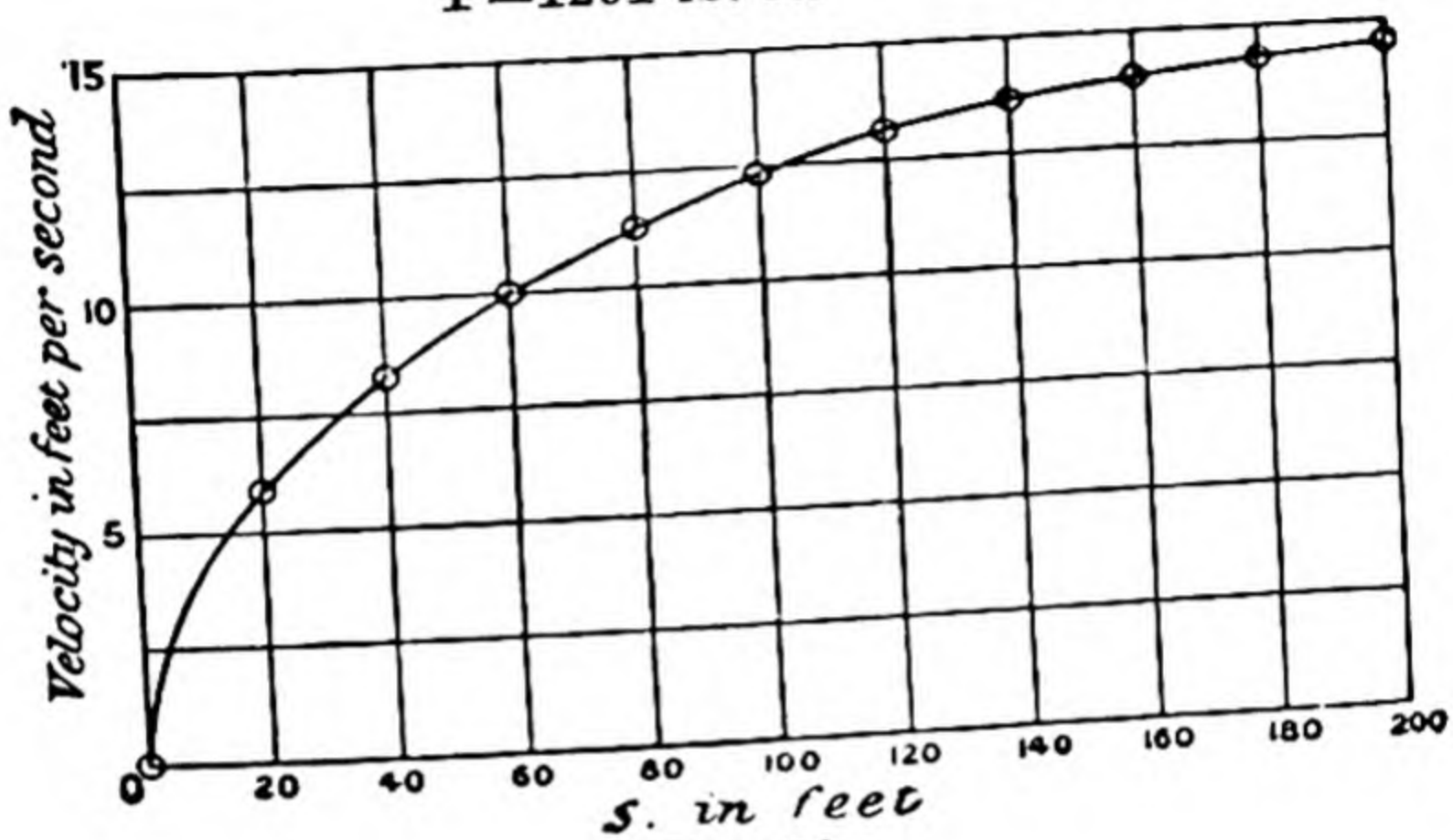

FIG. 78.

Time for 50 revs. at average speed of 90 r.p.m.

$$= 60 \times 50/90 = 33{\cdot}33 \text{ sec.}$$

$$\text{Average h.p.} = 405{,}500 \text{ ft.-lb.}/(33{\cdot}33 \text{ sec.} \times 550 \text{ ft.-lb./sec.})$$

$$= 22{\cdot}14.$$

$$\text{Maximum h.p.} = 2 \times 22{\cdot}14 = 44{\cdot}3.$$

Note that we could also solve this from the rate of change of angular momentum; taking the angular acceleration from (8), Art. 64, as $\alpha = \omega^2/2\theta = 36\pi^2/(2 \times 2\pi \times 50) = 9\pi/50$ radian/sec.^{2}

$$\text{Then } T = \alpha I = \frac{9\pi}{50} \times \frac{6000 \times 3{\cdot}5^2}{32{\cdot}2} = 1291 \text{ lb.-ft. as above.}$$

65. Inertia of Rolling Bodies (Wheels). So far, motion of rotation about an axis and translation in a straight line for a single body have been dealt with separately, both in considering rate of change of momentum in Art. 58 and K.E. in Art. 64. But when a body rolls on a track it has

motion of both kinds. Instantaneously, it is turning about an axis through the parts in contact with the track. For dynamical purposes, the motion is equivalent to that of the whole mass at the c.g. together with a rotation about an axis through the c.g. If v is the velocity of the c.g. of a wheel or a solid of revolution and r the distance of the c.g. from the track, the angular velocity of the wheel or other rolling body is

$$\omega = v/r \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The total K.E. is

$$\tfrac{1}{2}\mathrm{W}v^2/g + \tfrac{1}{2}\mathrm{W}k^2\omega^2/g \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where k is the radius of gyration about the axis of spin. This may also be written

$$\text{K.E.} = \tfrac{1}{2}(\mathrm{W}v^2/g)\{1 + k^2/r^2\} \quad . \quad . \quad . \quad (3)$$

Thus the inertia of the body in rolling is increased $1 + k^2/r^2$ times that of the same body sliding along at the same speed without rotation. For example, in a solid disc of uniform thickness and material, rolling on its edge (or a cylinder rolling on its curved surface), since $k^2/r^2 = 0{\cdot}5$, the inertia is increased 50 per cent. by rolling. In a very thin circular hoop without spokes or hub, the increase would approximate to 100 per cent. since k approaches r.

Note that k may exceed r in the case of a wheel mounted on a spindle which rolls on a track while the wheel overhangs it or lies between two parts of it.

The inertia of wheeled vehicles is increased by the rotary inertia of the wheels, but often the wheels are so small in proportion of the total moving weight that the rotary inertia of the wheels is negligible. But to take account of it, let W_1 be the total weight of a loaded vehicle including, say, W_2 the weight of the wheels which have a radius of gyration k and radius r from axis to track. Then from (3) the total K.E. is $\tfrac{1}{2}\mathrm{W}_1v^2\{1 + (\mathrm{W}_2/\mathrm{W}_1)(k^2/r^2)\}$. Thus if the wheels amounted to, say, 5 per cent. of the total weight (or $\mathrm{W}_2/\mathrm{W}_1 = 0{\cdot}05$) and the radius of gyration were $0{\cdot}8r$ or $k^2/r^2 = (0{\cdot}8)^2 = 0{\cdot}64$, the total K.E. would be $1 + 0{\cdot}05 \times 0{\cdot}64$ or $1{\cdot}032$ times that of the same weight moving with the speed of the c.g., and the error in neglecting the rotary inertia would be 3·1 per cent.

Example. Find the speed attained by a solid uniform circular disc or cylinder in rolling with axis horizontal down an incline with a total height h.

By the principle of Conservation of Energy, if W is weight of the body,

$$Wh=\tfrac{1}{2}(Wv^2+Wk^2\omega^2)/g=\tfrac{1}{2}Wv^2(1+k^2/r^2)/g$$

and for a solid cylinder or disc $k^2/r^2=0{\cdot}5$, hence

$$v^2=2gh/(1+k^2/r^2)=4gh/3$$

$$v=\sqrt{4gh/3}$$ (instead of $\sqrt{2gh}$ for a sliding body).

From this it is evident, since $v^2=2fs$, if s is h . cosec α for a slope at angle α to the horizontal, that

$$2fs=2gh/(1+k^2/r^2)$$

and $$f=g(h/s)/(1+k^2r^2),$$ or, $g \sin \alpha/(1+k^2/r^2)$.

These results may easily be obtained by considering the forces and moment exerted on the body rolling down

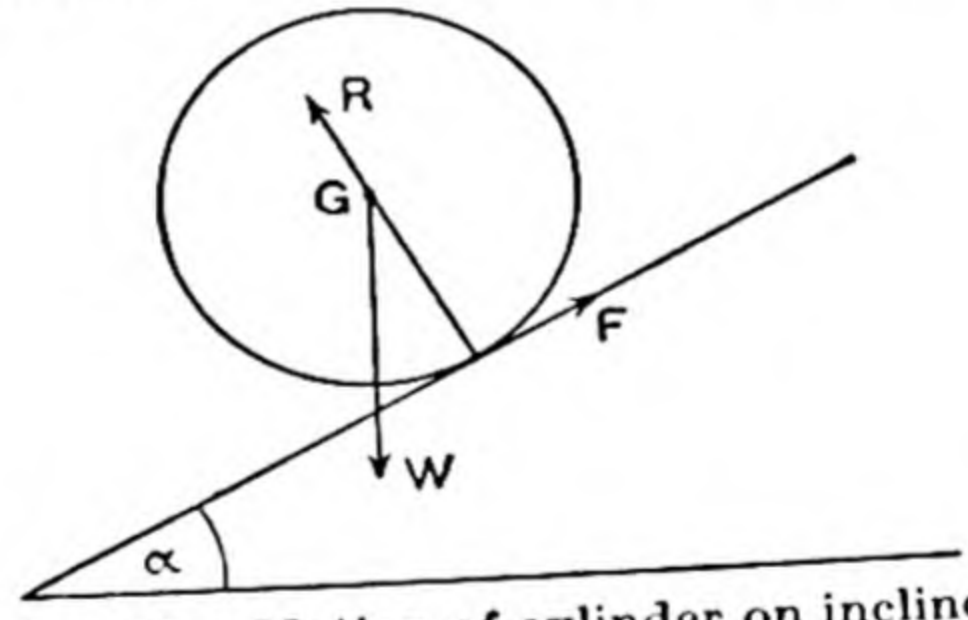

Fig. 79.—Motion of cylinder on incline.

the incline. The forces are shown in Fig. 79. F is the tangential thrust and R the normal thrust of the track on the wheel. In the (downward) direction of motion, the effective accelerating force,

$$W \sin \alpha - F = Wf/g$$

And about a horizontal axis through G, perpendicular to the diagram, the moment giving an angular acceleration (f/r), is Fr, hence

$$F \,.\, r=(Wk^2/g)\times(f/r)$$

$$F=W(k^2/r^2)(f/g)$$

and substituting this in the equation $W \sin \alpha - F = Wf/g$,

$$f/g\{1+k^2/r^2\} = \sin \alpha$$

$$f = g \sin \alpha/(1+k^2/r^2) \text{ as before.}$$

Notice that the first solution easily and quickly gives v but no information about F.

The value of F is $W(k^2/r^2)(f/g)$, which is equal to $(W \cdot \sin \alpha)k^2/(r^2+k^2)$, but the greatest value F can have is $\mu R = \mu . W \cos \alpha$, where μ is the coefficient of friction between the rolling body and the track. Hence $(W \sin \alpha)k^2/(r^2+k^2)$ must not exceed $\mu W \cos \alpha$ or $\tan \alpha$ must not exceed $\{1+(r^2/k^2)\}\mu$ if there is to be no slipping or sliding but pure rolling.

66. Alternative Solutions of Dynamical Problems. In this chapter, problems have been solved by applying the second law of motion and stating the equality between the rate of change of momentum and the effective (or unbalanced) forces in the direction of the acceleration. Or for rotary motion, by equating the rate of change of moment of momentum to the effective torque. Alternatively the time-integrals of both sides of such equations have been used, *i.e.* the impulse of the effective forces has been equated to the total corresponding change of momentum. (And in rotary motion, the angular impulse has been equated to the total change of moment of momentum.)

But, in many problems, yet another method is applicable, namely that of energy. The work done by the effective forces in accelerating a body may be equated to the increase in K.E. produced in the body. Since the algebraic expressions for K.E. arise directly from those for forces and movements, the methods are not mathematically independent. The choice of method is to some extent one of individual preference. The different equations used represent different physical principles, *e.g.* on the one hand the second law of motion and on the other the conservation of energy. Where the data allow the work done on, or energy given to, a body to be directly calculated, the speed or change in speed may conveniently and simply be calculated from the energy equation without determining the time taken or the acceleration produced. But as was

noted in the example at the end of Art. 65, it does not directly give the forces involved and which may be required. We now proceed to illustrate this type of solution of problems by the energy equations by examples previously solved by other means, pointing out the advantages and disadvantages according to circumstances. To understand mechanics, each type of solution should be considered, but in application to engineering problems it is desirable to have the ability to choose the method most direct and simple for a particular problem, *i.e.* for given data and for finding a particular quantity when others are not required. Since the expressions for K.E. involve the square of a speed, the energy equation is well adapted for finding a speed from the energy expended in producing it. The three methods, set out in algebraic form with the symbols of preceding articles, are as follows, for linear motion with constant acceleration under a constant force P.

(1) Effective force = mass × acceleration

$$P = mf \text{ or } Wf/g$$

(2) Impulse = gain of momentum

$$Pt = mft = m(v_2 - v_1) \text{ or } (W/g)(v_2 - v_1)$$

(3) Work done = gain of K.E.

$$Ps = \tfrac{1}{2}m(v_2^2 - v_1^2) \text{ or } \tfrac{1}{2}(W/g)(v_2^2 - v_1^2)$$

EXAMPLE 1. The example 2 of Art. 58 might proceed thus:

$$3227 \text{ lb.} \times s = \tfrac{1}{2}\{200 \times 2240 \text{ lb.}/32{\cdot}2 \text{ (ft./sec.}^2)\} \times (36{\cdot}7)^2 (\text{ft./sec.})^2$$

$$s = 2903 \text{ ft.}$$

$$t = 2903 \text{ ft.}/(\tfrac{1}{2} \times 36{\cdot}7) \text{ ft./sec.} = 158 \text{ seconds.}$$

This solution offers no advantage over that in Art. 58 and merely illustrates an alternative method.

EXAMPLE 2. Example 4 of Art. 58 might be solved as follows:—

$$\tfrac{1}{2}I\omega^2 = \tfrac{1}{2} \cdot 1553 \text{ lb.-ft. sec.}^2 \times \omega^2 = 200 \text{ lb.-ft.} \times 60 \text{ sec.} \times \tfrac{1}{2}\omega$$

$$\omega = 200 \times 60/1553 \text{ radians/sec.}$$

$$= 7{\cdot}73 \text{ radians/sec.}$$

Again the energy solution offers no advantage but is an alternative to that in Art. 58.

EXAMPLE 3. Find the speed of the two weights in Fig. 72, Art. 61, after they have moved a vertical distance h and hence find the acceleration.

The equation of energy is

$$\tfrac{1}{2}(W_1+W_2)v^2/g=(W_1-W_2)h$$

$$v^2=2gh(W_1-W_2)/(W_1+W_2)$$

$$v=\sqrt{2gh(W_1-W_2)/(W_1+W_2)}$$

And since also $v^2=2fh$

$$f=g\times(W_1-W_2)/(W_1+W_2)$$

The solution does not give P (which is $W_1(1-f/g)$).

EXAMPLE 4. Example 1 of Art. 62, involving the inertia of linear motion with that of rotation, can well be solved by an equation of energy so far as concerns speed and acceleration. The equation of energy for a vertical displacement h of W_1 is

$$\tfrac{1}{2}(W_1v^2+W_2k^2\omega^2)/g=W_1h$$

$$(W_1v^2+W_2k^2v^2/r^2)=2W_1gh$$

$$v^2=2gh/\{1+(W_2/W_1)(k^2/r^2)\}$$

$$v=\sqrt{2gh/\{1+(W_2/W_1)(k^2/r^2)\}}.$$

And since $v^2=2fh$

$$f=g/\{1+(W_2/W_1)(k^2/r^2)\}$$

As in (4), Art. 62, and $\alpha=f/r$ as in (5), Art. 62. This solution is simple for v; the tension P may be found from (3), Art. 62.

EXAMPLE 5. Similarly, Example 2 of Art. 62 may be solved by equating the K.E. to the net work done in producing acceleration. Taking an angular displacement of

the shaft equal to θ, the vertical displacement of W_1 is $r_1\theta$ and of W_2 is $r_2\theta$. The equation of energy is

$$\tfrac{1}{2}(W_3k^2\omega^2+W_1\omega^2r_1^2+W_2\omega^2r_2^2)/g=(W_1r_1-W_2r_2)\theta$$

ωr_1 and ωr_2 being the speeds of W_1 and W_2 respectively,

$$\omega^2=2g\theta(W_1r_1-W_2r_2)/(W_3k^2+W_1r_1^2+W_2r_2^2)$$

And since $\omega^2=2a\theta$ ((8), Art. 64)

$$a=g(W_1r_1-W_2r_2)/(W_3k^2+W_1r_1^2+W_2r_2^2)$$

as in (4), Art. 62. This method of solution is particularly simple and direct for finding ω, and hence the speed of the weights (viz. ωr_1 and ωr_2), but it does not give the tensions P_1 and P_2 which need the equations of force. If these forces are required, the direct application of the second law of motion (Art. 62) might well be used for the solution of the whole problem.

In this and preceding problems it may be noticed (as guidance for others) that the solution by the energy equation is simple and direct for certain purposes (finding speed and acceleration), but does not furnish the information about the forces involved afforded by the methods used in Art. 62, *i.e.* the application of the second law of motion to each moving mass, thus forming equations of *forces*. Nevertheless, the forces are easily found when the acceleration is known. Thus P_1 and P_2 are easily found from (10) and (11) of Art. 62 when a has been determined, as above, from the energy equation.

EXAMPLE 6. In a lifting machine the effort moves four times as fast as the load. (*a*) If friction and the inertia of the machine are negligible and a weight of 20 lb., used as an effort, lifts a load of 64 lb., find the K.E. and speed of the load and effort, after the effort has moved 10 feet vertically downwards. (*b*) How would the results be modified if 10 per cent. of the energy used were absorbed in friction, *i.e.* if the efficiency were 90 per cent.? (*c*) What would be the effect of the inertia of a drum from which is unwound a cord which carries the 20 lb. driving weight?

(*a*) Energy input $=20$ lb. $\times 10$ ft. $=200$ ft.-lb.

Energy used in lifting load $=64$ lb. $\times 10/4$ ft. $=160$ ft.-lb.

K.E. of moving masses (by difference) $=40$ ft.-lb.

Let v be the speed of effort; then $v/4$ = speed of load.

$$\text{K.E.} = \tfrac{1}{2} \times 20v^2/32{\cdot}2 + \tfrac{1}{2} \times 64(\tfrac{1}{4}v)^2/32{\cdot}2 = 40 \text{ ft.-lb.}$$

$$v^2(20+4) \text{ lb.} = 40 \times 64{\cdot}4 \text{(ft.-lb.)} \times \text{ft./sec.}^2$$

$$v^2 = 40 \times 64{\cdot}4/24 = 107{\cdot}3 \text{(ft./sec.)}^2$$

$$v = 10{\cdot}35 \text{ ft./sec}$$

Speed of load $= 10{\cdot}35/4 = 2{\cdot}59$ ft./sec.

From the above it is evident that the K.E. of 40 ft.-lb. is divided between the effort and load masses in the ratio 20 to 4 or 20/24 of $40 = 33{\cdot}67$ ft.-lb. in the 20 lb. effort weight and 6·33 ft.-lb. in the 64 lb. load.

Since the 20 lb. mass attains a speed of 10·35 ft./sec. in 10 ft. travel, the acceleration ($v^2/2s$) is $107{\cdot}3\text{(ft./sec.)}^2 \div 20$ ft. $= 5{\cdot}36$ ft./sec.^{2}, and the acceleration of the 64 lb. mass is $5{\cdot}36/4 = 1{\cdot}34$ ft./sec.^{2}

The tension of the cord carrying the descending 20 lb. mass is $20 - 20 \times 5{\cdot}36/32{\cdot}2 = 20\ (1 - 0{\cdot}166) = 16{\cdot}7$ lb. and the tension of the lifting rope is $64(1 + 1{\cdot}34/32{\cdot}2) = 64(1{\cdot}0416) = 66{\cdot}7$ lb.

(*b*) If 10 per cent. of the energy input were lost in friction the useful work done would be 180 ft.-lb. instead of 200 ft.-lb., and subtracting the work done in lifting (160 ft.-lb), the K.E. of the moving weights would give 20 ft.-lb. instead of 40 ft.-lb., v^2 would be reduced 50 per cent. and v would be $1/\sqrt{2} = 0{\cdot}707$ times its previous value (*i.e.* reduced nearly 30 per cent.) and the acceleration would be reduced 50 per cent., since, *for a given travel* it is proportional to v^2 ($f = v^2/2s$).

(*c*) The inertia of the drum would add to the left side of the above energy equation a term $\frac{1}{2}Wk^2\omega^2/g$ where W is the weight of the drum and ω its angular velocity. The quantity $k^2\omega^2$ could easily be expressed in terms of v^2 from the dimensions of the drum, for if r is the effective radius of the drum, $\omega r = v$ or $\omega = v/r$, $k\omega = vk/r$. The equation would then be solved for v^2 as before and would show a

diminished value. (The driving torque remains as before but the inertia is increased.)

EXAMPLE 7. A shaft A is geared to a shaft B which it drives at four times its own speed. Shaft A and a wheel attached to it weigh 50 lb. and have a radius of gyration of 18 inches; B and an attached wheel weigh 30 lb. and have a radius of gyration of 12 inches. Find the K.E. of the system when shaft A reaches a speed of 200 r.p.m., and the proportion of the whole in A and B respectively. Find the constant torque which would, if exerted on A, bring the system up to full speed in 50 revolutions of A.

$$\omega_A = 2\pi \times 200/60 = 6{\cdot}67\pi \text{ radians/sec.},$$

$$\omega_B = 26{\cdot}67\pi \text{ rad./sec.}$$

$$\text{K.E.} = \tfrac{1}{2} \cdot 50 \times (1{\cdot}5)^2 \times (6{\cdot}67\pi)^2/32{\cdot}2 + \tfrac{1}{2} \cdot 30 \times 1^2 \times (26{\cdot}67\pi)^2/32{\cdot}2 \text{ ft.-lb.}$$

$$= (6{\cdot}67\pi)^2(50 \times 2{\cdot}25 + 30 \times 16)/64{\cdot}4 \text{ ft.-lb.}$$

$$= 6{\cdot}813(112{\cdot}5 + 480) = 4036 \text{ ft.-lb.}$$

Proportion of K.E. in A $= 112{\cdot}5/592{\cdot}5 = 19$ per cent.
Proportion of K.E. in B $= 480/592{\cdot}5 = 81$,, ,,

Let T be the torque on A. Then in 50 revolutions or 100π radians

$$T \times 100\pi = 4036 \text{ ft.-lb.}$$

$$T = 40{\cdot}36/\pi = 12{\cdot}84 \text{ lb.-ft.}$$

The torque might alternatively be found from the angular acceleration. With constant torque, the average speed in running up to 200 r.p.m. will be 100 r.p.m., and therefore the time taken to make the 50 revolutions will be 30 seconds and the angular acceleration of A will be $200 \times 2\pi/(60 \times 30) = 2\pi/9$ radians/sec.^{2}, and that of B will be four times this. Also, to give any particular torque to B will require four times that torque on A. Hence

$$T = \{50 \times 1{\cdot}5^2 \times (2\pi/9) + 4 \times 30 \times 1^2 \times 8\pi/9\}/32{\cdot}2$$

$$= 2\pi(112{\cdot}5 + 480)/(32{\cdot}2 \times 9) = 12{\cdot}84 \text{ lb.-ft. as before.}$$

67. Changes in Kinetic Energy and Speed. Flywheels. A change of speed of a moving body involves a change in its K.E. When the speed of a moving body is diminished, the body gives up some energy; and when it is accelerated, it absorbs some energy. Thus a fast moving body of considerable mass may act as a temporary reservoir, storing and restoring energy, particularly during a succession of events which is repeated at intervals as in the cycle of operations in a reciprocating heat engine—either a steam or internal combustion engine. Thus a flywheel is used as a means of reducing speed fluctuations, within a cycle of events, in machines of various kinds. It decelerates and gives up energy when otherwise the machine, of which it virtually forms a part, would be unduly reduced in speed owing to a deficiency of driving effort or an excess of resistance or load at some part of a cycle of events. It is then accelerated in periods when the driving effort is in excess of that required for driving the machine. Thus the speed is not made uniform but the cyclical changes in speed (*e.g.* in one revolution) are reduced and controlled. The heavier the flywheel and the higher the speed, the smaller the speed variations and we now consider the relation of the inertia and speed of the wheel to speed variation.

If ω_1 is the maximum and ω_2 the minimum angular velocity of a wheel of weight W and radius of gyration k the fluctuation of K.E. from the maximum to the minimum, expressed as a fraction of the K.E. at the mean speed, $\frac{1}{2}(\omega_1+\omega_2)$ is

$$\left.\begin{aligned}\tfrac{1}{2}\mathrm{W}k^2(\omega_1{}^2-\omega_2{}^2)/\tfrac{1}{2}\mathrm{W}k^2(\tfrac{1}{2})^2(\omega_1+\omega_2)^2\\ =2(\omega_1-\omega_2)/\{\tfrac{1}{2}(\omega_1+\omega_2)\}\end{aligned}\right\} \quad . \quad . \quad (1)$$

i.e. twice the extreme fluctuation of speed expressed as a fraction of the mean speed. Or, again, if the speed increases one per cent. the proportional gain of K.E. is

$$\{(1{\cdot}01\omega)^2-\omega^2\}/\omega^2=0{\cdot}02 \quad . \quad . \quad . \quad . \quad (2)$$

(or 2 per cent.) very nearly.

If, in order to find a suitable flywheel for a given machine, we plot graphs of the driving torque and the resisting torque on angular displacement of the shaft as a base,

both will have the same mean height (*i.e.* the torques will have the same mean values), but the two curves will generally cross one another forming loops. The areas of the loops (between the two graphs) will represent energy to be stored by a flywheel in acceleration when the driving torque is in excess of the resisting torque, and energy to be restored by the flywheel when, during deceleration, the resisting torque exceeds the driving torque. The geometrical form of these graphs and loops will depend upon the variations in driving effort and in the resistances. Suppose the excess energy alternately stored and returned, to be δE, say, in ft.-lb. Then the extreme fluctuation in energy from a maximum speed ω_1 to a minimum speed ω_2 is δE or

$$\tfrac{1}{2}Wk^2(\omega_1^2-\omega_2^2)/g=\delta E \quad . \quad . \quad . \quad . \quad (3)$$

where W is the weight of the flywheel and k is its radius of gyration.

Hence $$\tfrac{1}{2}Wk^2(\omega_1+\omega_2)(\omega_1-\omega_2)/g=\delta E \quad . \quad . \quad . \quad (4)$$

Now suppose the speed fluctuation from highest to lowest not to exceed n times the mean speed, where n is normally a small fraction such as 0·01 say, then

$$\omega_1-\omega_2=n\times\tfrac{1}{2}(\omega_1+\omega_2) \quad . \quad . \quad . \quad . \quad (5)$$

And $$(\omega_1^2-\omega_2^2)=(\omega_1-\omega_2)(\omega_1+\omega_2)$$
$$=n\times\tfrac{1}{2}(\omega_1+\omega_2)^2=2n\{\tfrac{1}{2}(\omega_1+\omega_2)\}^2 \quad . \quad (6)$$

Substituting this in (3)

$$\tfrac{1}{2}Wk^2\times 2n\{\tfrac{1}{2}(\omega_1+\omega_2)\}^2/g=\delta E \quad . \quad . \quad . \quad (7)$$

$$Wk^2=g\delta E/n\bar{\omega}^2 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $\bar{\omega}$ is the mean speed $\tfrac{1}{2}(\omega_1+\omega_2)$. When all the quantities g, δE, and $\bar{\omega}$ are given, the product Wk^2 is known; and if k is decided upon, W can be simply found.

EXAMPLE. A single-cylinder internal-combustion 4-stroke engine of 20 i.h.p. runs at 250 r.p.m., and between the minimum and maximum speeds the energy stored is three times that developed per revolution. If the maximum speed fluctuation (from minimum to maximum) is to be limited to 1·5 per cent. of the mean speed, find what

weight of flywheel will be required if its radius of gyration is 3 ft.

Work per rev. $=20\times33{,}000/250$ ft.-lb.
$=2640$ ft.-lb.

Work stored between minimum and maximum speed
$=\frac{1}{2}Wk^2(\omega_1^2-\omega_2^2)/g=3\times2640=7920$ ft.-lb.

And $\omega_1-\omega_2=0{\cdot}015\times\frac{1}{2}(\omega_1+\omega_2)$

$\omega_1^2-\omega_2^2=0{\cdot}015\times\frac{1}{2}(\omega_1+\omega_2)^2$

$=0{\cdot}03\times\{\frac{1}{2}(\omega_1+\omega_2)\}^2$

$=0{\cdot}03\times(250\times2\pi/60)^2$

$=0{\cdot}03\times(26{\cdot}18)^2$

$=20{\cdot}56/(\text{radians/sec.})^2$

$$Wk^2=\frac{2\times32{\cdot}2\text{ ft.}}{\text{sec.}^2}\times\frac{7920\text{ ft.-lb.}\times\text{sec.}^2}{20{\cdot}56}=24{,}800\text{ lb.-ft.}^2$$

And if $k^2=9$ ft.^{2}, $W=24{,}800/9=2767$ lb.

EXAMPLES V

1. A train weighing 200 tons has a resistance of 15 lb. per ton, supposed constant at all speeds. What tractive force will be required to give it a speed of 30 miles per hour in 1·5 minutes ?

2. A train travelling at 40 miles per hour is brought to rest by a uniform resisting force in half a mile. How much is the total resisting force in pounds per ton ?

3. A gun weighing 40 tons projects a shell weighing 1500 lb. with a velocity of 1400 feet per second. With what initial velocity will the gun recoil ? What average force will be required to bring it to rest in 3 feet ?

4. A gun weighing 40 tons has its velocity of recoil destroyed in 2 feet 9 inches by an average force of 70 tons. If the projectile weighed 14 cwt., find its initial velocity.

5. A lift has an upward acceleration of 3·22 feet per second per second. What pressure will a man weighing 140 lb. exert on the floor of the lift ? What pressure would he exert if the lift had an acceleration of 3·22 feet per second per second downward ? What upward acceleration would cause his weight to exert a pressure of 170 lb. on the floor ?

6. A body is to be moved up a track inclined 30° to the horizontal by a horizontal force and is to have an acceleration

equal to half that of a body falling freely. If the body weighs 80 lb. and the coefficient of friction between it and the track is 0·3, find the force necessary.

7. A bicycle has wheel centres 4 ft. apart. The cycle and rider together weigh 180 lb., and combined have a c.g. 3 feet above ground and 1 foot in front of the rear axle. Find the pressures between each wheel and the ground (*a*) when the machine is moving with constant speed, and (*b*) when subject to a deceleration of 8 ft./sec.^{2} due to a brake on the front wheel.

8. A flywheel weighing 1·5 tons has a radius of gyration of 4 feet. If it attains a speed of 80 revolutions per minute in 40 seconds, find the mean effective torque exerted upon it in pound-feet ?

9. A weight of 40 lb. attached to a cord which is wrapped round the 2-inch spindle of a flywheel descends, and thereby causes the wheel to rotate. If the weight descends 6 feet in 10 seconds, and the friction of the bearing is equivalent to a force of 3 lb. at the circumference of the spindle, find the moment of inertia of the flywheel. If it weighs 212 lb., what is its radius of gyration ?

10. If the weight in Question 9, after descending 6 feet, is suddenly released, how many rotations will the wheel make before coming to rest ?

11. A flywheel weighing 250 lb. is mounted on a spindle 2·5 inches diameter, and is caused to rotate by a falling weight of 50 lb. attached to a string wrapped round the spindle. After falling 5 feet in 8 seconds the weight is detached, and the wheel subsequently makes 100 revolutions before coming to rest. Assuming the tangential frictional resisting force at the circumference of the axle to be constant throughout the accelerating and stopping periods, find the radius of gyration of the wheel.

12. Find the acceleration and tensions on the two ends of the cord in Fig. 72, if the radius of the pulley is r and its radius of gyration k and its inertia is not negligible, its weight being W_2.

13. A load of 300 lb. is lifted by a cable wrapped round a drum 18 inches diameter having a radius of gyration of 8 inches and weighing 500 lb. If the load has an upward acceleration of 0·5 ft./sec.^{2}, find the necessary torque on the drum.

14. A car weighs 2000 lb. What is its K.E. in ft.-lb. when running at 40 m.p.h.? If the resistances amount to 50 lb., how far will it run with the motor disengaged (*a*) on the level, (*b*) up a slope of 1 in 40 ?

15. A flywheel is required to store 8000 ft.-lb. of energy when its speed increases from 249 to 251 revolutions per minute. What would be its K.E. at 250 revolutions per minute and if its radius of gyration is to be 3 feet, what should it weigh ?

16. What uniform torque will be required to bring the flywheel in Question 15 to rest in 80 revolutions ?

17. A wheel, 6 feet diameter, with a radius of gyration of 2 feet 9 inches, rolls down a slope of 1 in 100. What speed will its centre acquire in running down a distance of 150 feet?

18. In an experimental wheel and axle, the effort is provided by a weight of 10 lb. suspended by a cord wrapped round the rim of the wheel which has an effective diameter of 2 feet. The load is lifted by a rope which is wound round a drum of 6 inches effective diameter. This effort will just lift 35 lb. without acceleration. If a load of only 30 lb. is lifted, find the total K.E. and the speed attained by the 10 lb. effort weight after it has traversed a distance of 6 ft. vertically downwards, assuming friction remains as when 35 lb. was lifted without acceleration. (*a*) If the rotary inertia is negligible. (*b*) If the rotary parts weigh 15 lb. and have a radius of gyration of 6 inches.

CHAPTER VI

CIRCULAR MOTION

68. Motion in a Circle. A common instance of motion in a curved path is motion in a circle and at constant speed. Note that while the speed may be constant, the velocity is not, for velocity has the characteristic of direction as well as magnitude; and since the direction is continually changing, the velocity is changing and there is acceleration, which also has direction. We have investigated the acceleration in Art. 50 and found it to be of magnitude $\omega^2 r$ or v^2/r and directed radially inwards toward the centre of the circle, where r is the radius, v the constant speed and ω (or v/r) the constant angular velocity. In order to have this acceleration, a moving body, in accordance with the second law of motion (Art. 58), must have exerted upon it a force proportional to its mass multiplied by the acceleration and in the direction of the acceleration, *i.e.* radially inward. If W is the weight of the body, the radial inward force exerted upon it must be

$$(W/g) \times \omega^2 r \quad \text{or} \quad (W/g) \times (v^2/r) = W v^2/gr \quad . \quad (1)$$

This is called the *centripetal* force acting on the body. We have seen that forces are dual; the body will exert a radially outward force, of the same magnitude as the centripetal force, on whatever constrains it to follow a circular path. This outward force is called the *centrifugal* force; it is exerted *by the body* on its constraints and not on the body. The magnitude of the centrifugal force is then

$$(W/g)\omega^2 r \quad \text{or} \quad W v^2/gr \quad . \quad . \quad . \quad . \quad (2)$$

(The solution of problems on bodies moving in circular orbits by d'Alembert's Principle (Art. 58) in the form of the introduction of a fictitious outward force on the body equal and opposite to the actual centripetal force has led to serious misconceptions; probably the conditions, *e.g.* the artifice of introducing a non-existent force, have been

omitted or forgotten. There is no need for this artifice, or this method, which proves so misleading.)

While there is no external radially outward force acting on a body moving in a circular path, parts of the body constrained by other parts so to move may and do exert outward (internal) forces on other parts, the material of the body being thus in a state of stress. Similarly, a truck moving in a curved track will have exerted upon it an outward force by a load which it carries, and the truck then exerts an equal inward force on its load. But these forces are internal and the joint system of truck and load will be subject to an inward force, *e.g.* from rails.

Conical Pendulum. One of the simplest examples of a body moving with constant speed in a circular path is

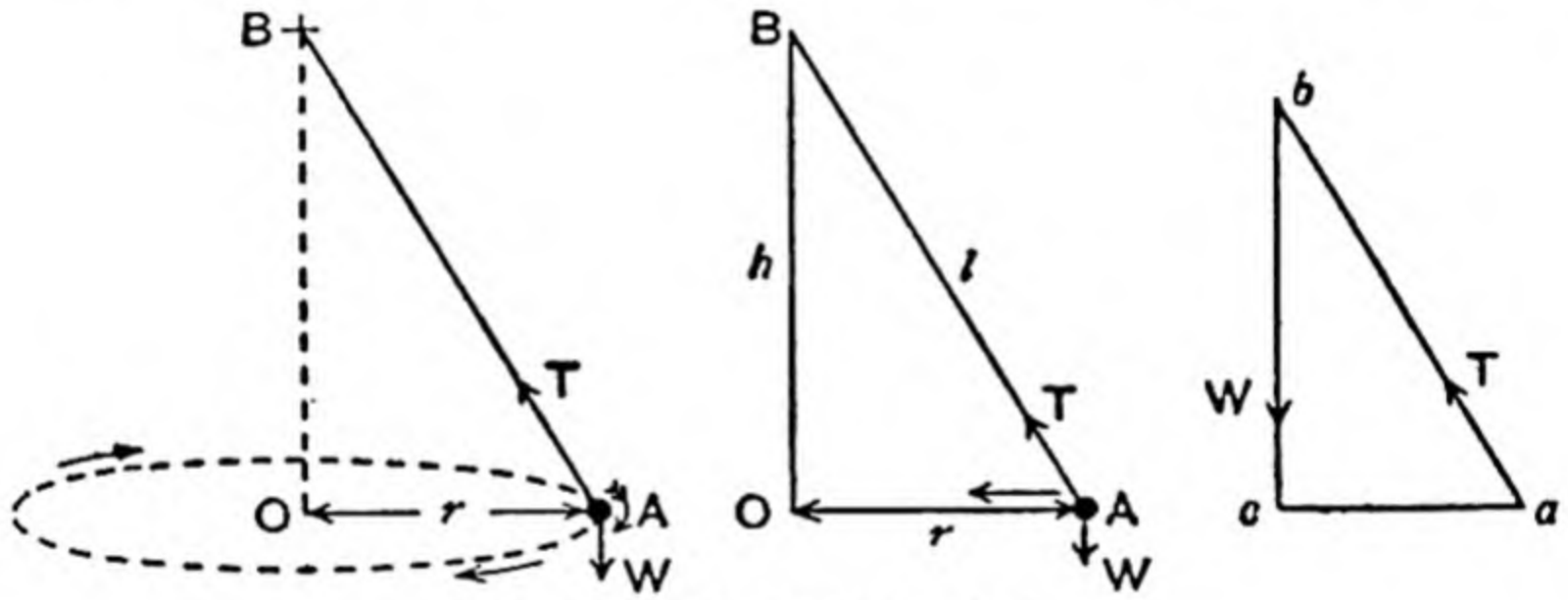

FIG. 80.—Conical pendulum.

that of a conical pendulum. This consists of a small but relatively heavy piece of material, generally a metal (called the bob of the pendulum and treated as a particle), suspended by a flexible thread AB from a fixed point B, Fig. 80, and moving with constant speed v or constant angular velocity ω, or v/r, in a circle of radius r. In such circumstances, if minor resistances are negligible, there are two forces only keeping the bob in its circular path and giving it the necessary acceleration. These are (1) the pull T of the thread AB, and (2) the gravitational pull W of the earth on the bob where W is its weight. These two forces must have a resultant along AO towards O and of magnitude Wv^2/gr (or $W\omega^2r/g$). The relations between T (the tension in the thread), the weight W and their

resultant $W\omega^2 r/g$ are shown in the vector triangle *abc* in Fig. 80, where *ab* is parallel to AB, *bc* is vertical and *ac* horizontal, and vectorially, $ac=ab+bc$. Then if l is the length of the thread AB and h is the vertical height (OB) of the conical pendulum,

$$\frac{ab}{bc}=\frac{T}{W}=\frac{l}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{W}{W\omega^2 r/g}=\frac{bc}{ac}=\frac{h}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

a result which may be found by equating the clockwise moment of W about B to the moment (also clockwise) of $W\omega^2 r/g$ about B, since the moment of a resultant is equal to the moment of its components and that of T about B is zero.

The height $$h=g/\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

is characteristic of the angular speed ω, is independent of l (for any given ω) and will be expressed in the units of length used for g. Thus if g is in ft./sec.^{2}, h will be in ft. If the speed ω is increased, h is reduced and the conical pendulum takes up a new shape with increased r since r remains equal to $\sqrt{l^2-h^2}$ where l is constant.

Instead of considering the forces exerted on the bob, we might have taken those acting on the thread AB. In so far as its mass is negligible, the forces exerted on it have no resultant, *i.e.* they are balanced or are in equilibrium. They are the pull in the direction AB exerted on the thread by its point of suspension at B and the force exerted on the thread by the bob. This may be split into two components: (1) W vertically downwards, and (2) the horizontal outward centrifugal force $W\omega^2 r/g$ which the bob exerts on its constraint, *i.e.* upon the thread. In this case the vector diagram will be a *closed* triangle *abc* ($ab+bc+ca=0$), whereas for the ball we had $ab+ba=ac$ (inward).

Centrifugal Governors. In engine governors the centrifugal pull which a revolving governor ball exerts on a suspension rod, or combination of rod and link, is resisted by some form of controlling force, either spring or gravitational. An increase of speed involves an increased centrifugal pull by the balls and a reduced height of the conical pendulum of which the balls are the bobs. The balls, in

pulling the links to a new position, thereby operate a lever (or series of levers) which controls and checks the supply of energy to the engine. In books dealing with governors the mechanics of the operation is very often misstated. As already indicated, there is no radial outward force exerted *on the ball*, nor is the ball in equilibrium ; it has a resultant force, directed radially inward, exerted upon it, giving it the radially inward acceleration of a body moving in a circular path. Governors are dealt with in Art. 70.

69. Motion on Curved Banked Tracks. Suppose a body P, Fig. 81, to be sliding round a smooth circular track of radius OP equal to r ft., with constant speed v. At what angle α to the horizontal plane should the track be inclined or banked in order that the body shall remain in its circular path ? The two forces exerted on the body are (1) its

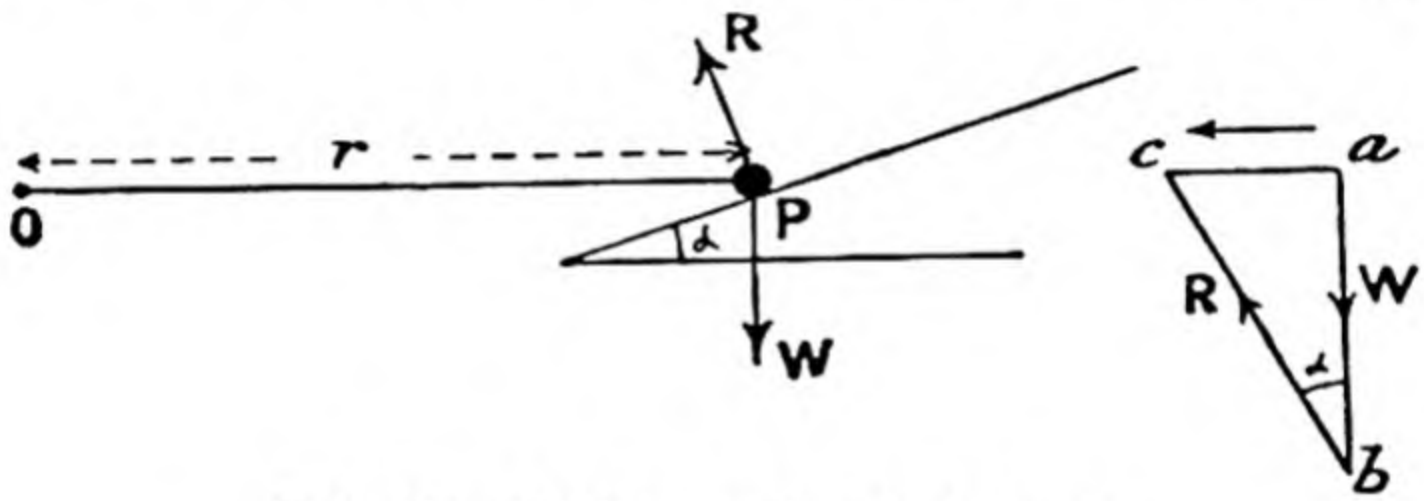

FIG. 81.—Motion on banked track.

weight W, and (2) the reaction R of the track, normal to the track, since this is assumed to be smooth and incapable of offering frictional resistance.

These two have a horizontal resultant Wv^2/gr towards the centre O of the horizontal circle in which the body moves. If we draw a vector, ab (Fig. 81), vertically, to represent W, then R is inclined at an angle α to it, where α is the angle of banking of the track. If a vector, bc, be drawn from b inclined at an angle α to ab, to meet ac, the perpendicular to ab from a, then bc represents R, and ac or $(ab+bc)$ represents the resultant of W and R, viz. Wv^2/gr and

$$\tan \alpha = \frac{ac}{ab} = \frac{W}{g} \cdot \frac{v^2}{r} \div W = \frac{v^2}{gr} \quad . \quad . \quad . \quad (1)$$

which gives the angle α required.

If a track is horizontal, or insufficiently banked, the radial inward or centripetal force exerted by the track on the moving body must be supplied by a component of the reaction of the track to the thrust of the body. In the case of a vehicle on a road, this radially inward force directed toward the centre of curvature of the track is frictional, while on a railway the force is exerted by the rail on the flanges of the wheels of the vehicle, while the centrifugal or outward thrust of the wheel flanges is exerted on the outer rail and would push it outwards but for the resistance exerted by the sleeper or tie embedded in suitable firm material.

In order to have the action and reaction normal to the track, the outer rail is raised, and the track thereby inclined to the horizontal. The amount of this "superelevation" suitable to a given speed is easily calculated.

Let d be the gauge in inches, say, v the velocity in feet per second, and r the radius of the curve in feet. Let AB (Fig. 82) represent d; then AC represents the height in

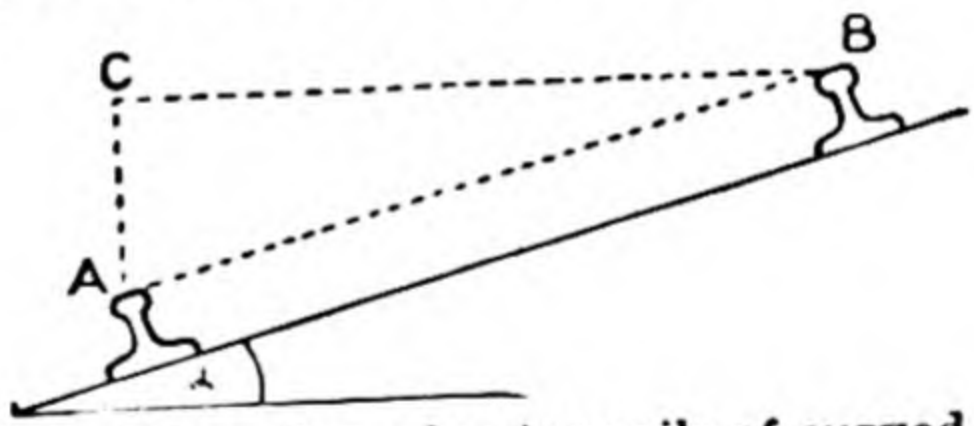

FIG. 82.—Elevation of outer rails of curved track.

inches (exaggerated) which B stands above A, and $A\hat{B}C$ is the angle of banking, α, as before. Then $AC = AB \sin \alpha$ $= AB \tan \alpha$ nearly, since α is always very small; hence AC represents $d \tan \alpha$, or, substituting for $\tan \alpha$ from (1), v^2d/gr inches.

Wheeled Vehicles. Since the forces which constrain a wheeled vehicle to keep in a circular path are approximately located by the contacts of the wheels and track, we can instructively investigate these forces. Let Fig. 83 represent a vehicle of weight W (symmetrical about a central plane perpendicular to Fig. 83), moving in a circle of radius r (centred to the left of the vehicle) on a level (*i.e.* unbanked) track, with speed v. The thrust of the vehicle

consists of two components: (1) the vertical downward thrust equal to the weight W of the vehicle, and (2) the horizontal centrifugal thrust Wv^2/gr, both through G the c.g. of the vehicle. These are shown by the arrows outward from G together with their resultant R which cuts the track AB in H. The forces exerted on the vehicle by the track are the reaction at A and B, the points of contact of the wheels with the track. They must have a resultant equal in magnitude to the thrust R of the vehicle on the track and in the same straight line, *i.e.* directed towards G in the line HG. Then the forces exerted on the vehicle (in or parallel to the plane of the diagram), are (1) a force R

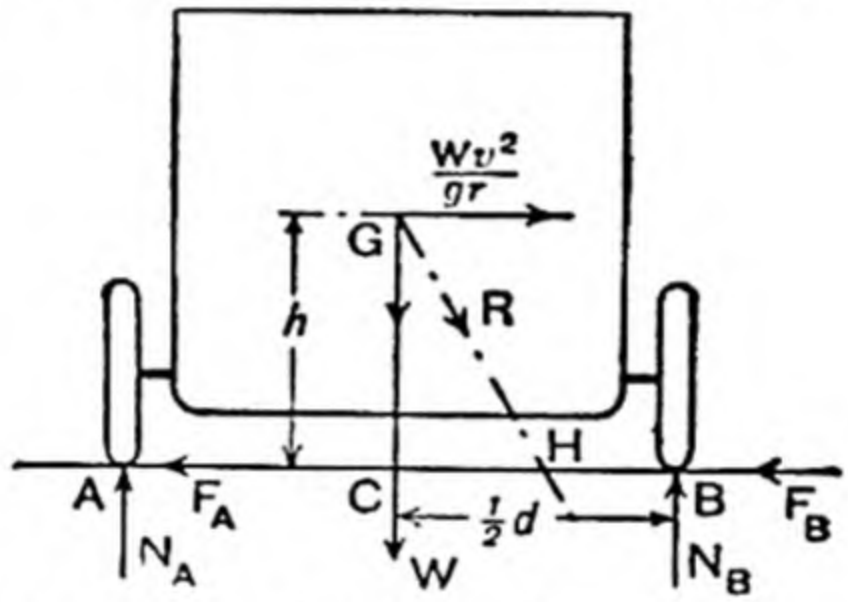

FIG. 83.—Forces exerted by and on vehicle on curved unbanked track.

in the position and direction HG, (2) the vertical pull of gravitation equal to W vertically downward through G, and these have a horizontal resultant Wv^2/gr, towards the left, through G. The point H is easily determined, for, either taking GCH as a vector triangle of forces, or from equation of moments of the components about H,

$$CH = GC \times (Wv^2/gr)/W = h \times v^2/gr \quad . \quad . \quad (2)$$

Splitting the reactions of the track at A and B into normal and lateral components N_A and F_A at A and N_B and F_B at B,

$$N_A + N_B = W \text{ and } F_A + F_B = Wv^2/gr \quad . \quad . \quad (3)$$

In the case of a rail track (except for friction which must be neglected in comparison with flange pressure), $F_A = 0$; and in the case of a road the lateral force will be limited to $\mu\ (N_A + N_B)$ where μ is the coefficient of friction

between the tyres and the road. We can find N_A and N_B conveniently by equating their moments about G to the moment of F_A+F_B (or Wv^2/gr) about G (since there is no rotation about G and the resultant force on the vehicle acts through it, with zero moment about it). Thus,

$$N_B \times \tfrac{1}{2}d - N_A \times \tfrac{1}{2}d = (Wv^2/gr) \times h \quad . \quad . \quad . \quad (4)$$

$$N_B - N_A = 2W(h/d) \times (v^2/gr)$$

and

$$N_B + N_A = W$$

Hence

$$\left.\begin{aligned} N_B &= W\left(\tfrac{1}{2} + \frac{h}{d}\cdot\frac{v^2}{gr}\right) \\ N_A &= W\left(\tfrac{1}{2} - \frac{h}{d}\cdot\frac{v^2}{gr}\right) \end{aligned}\right\} \quad . \quad . \quad (5)$$

the fraction hv^2/dgr representing the proportion of the vertical load W transferred from the inner to the outer rail at the speed v.

We may assume in the case of a road that, short of the slipping speed, $F_A/F_B = N_A/N_B$, *i.e.* that the two oblique or resultant reactions of the road on the vehicle are parallel. In the case of rails, $F_A = 0$ (conventionally at least) and $F_B = Wv^2/gr$, the reaction at B being inclined while that at A is vertical and is N_A. (The line of action of the reaction at B will pass through B and through the point of intersection of a vertical line through A with the line HG produced.)

At the limit of speed at which the vehicle is about to upset or rotate about an axis perpendicular to the diagram,[1] *i.e.* when H reaches B, $N_A = 0$, and $N_B = W$,

$$W \times \tfrac{1}{2}d = (Wv^2/gr) \times h \quad \text{or} \quad (h/d) \times \frac{v^2}{gr} = \tfrac{1}{2}$$

or

$$v^2 = gr \times d/(2h) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

[1] As v increases, N_B increases and N_A decreases and thus increase the moment, counter-clockwise, in Fig. 83, which balances the clockwise moment of $F_B + F_A$ about G. But a limit to this adjustment is reached when N_A reaches zero, since it cannot change sign and act downwards. At the limit, $N_B = W$ and the moment is $W \times \tfrac{1}{2}d$, balancing a moment $(Wv^2/gr) \times h$ about G. Subsequent rotation of the vehicle about a horizontal axis as F_B increases does not imply a *radial* outward movement of G, for

Unless h is large, *i.e.* the c.g. is high, side-slip will occur at a lower speed than v given by (6). For if F_A+F_B (*i.e.* Wv^2/gr) $=\mu(N_A+N_B)=\mu W$,

$$\mu=v^2/gr \text{ and } v^2=\mu gr \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which will be less than $grd/(2h)$ of (6) if μ is less than $\frac{1}{2}d/h$, *i.e.* unless h is less than $\frac{1}{2}d/\mu$; *e.g.* if $\mu=0{\cdot}5$, the speed given by (7) will be less than that given by (6) if h is less than d, as it would be in most road motor vehicles. The condition for which the vehicle is on the point of upsetting and

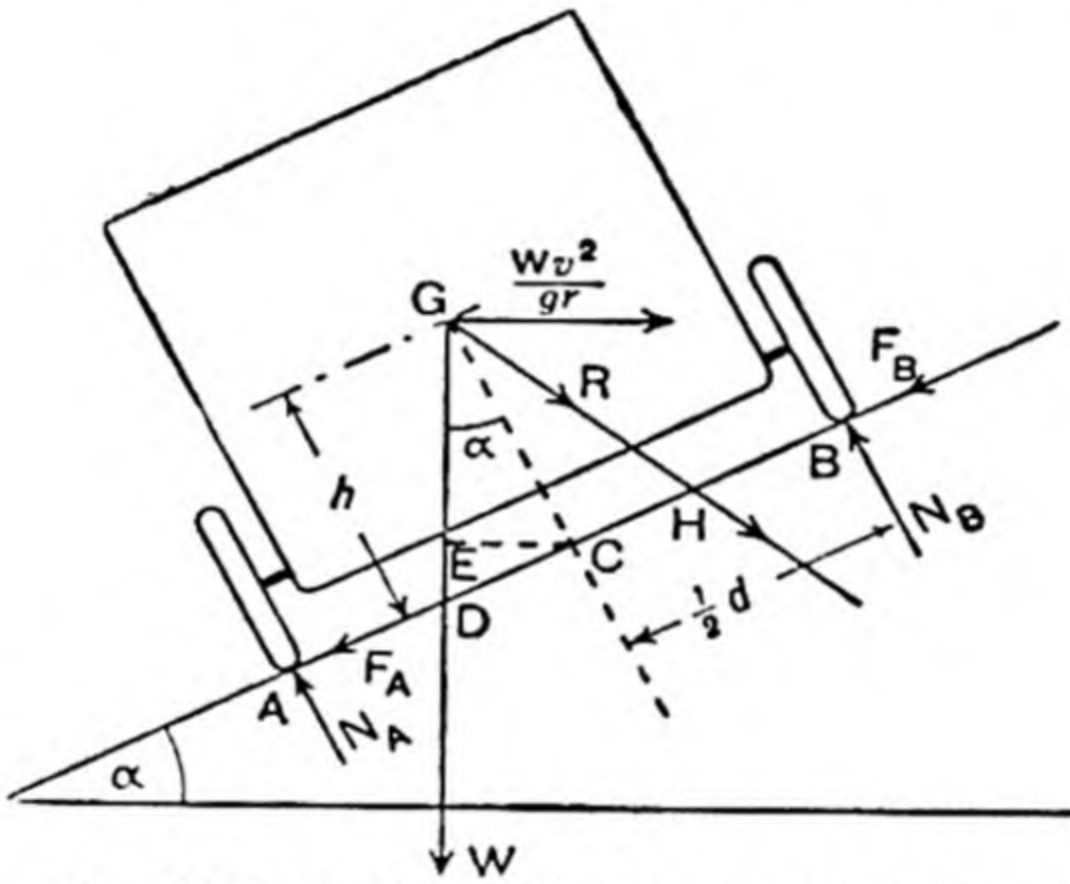

FIG. 84.—Motion of vehicle on curved banked track.

sliding is given by equating (6) and (7) and is stated in (15). In the case of a rail vehicle, the speed limit would remain as given by (6) for overturning, namely, $v^2=\frac{1}{2}grd/h$.

Proceeding to consider a similar symmetrical vehicle on a banked curve, Fig. 84, it is evident that at rest and at low

Fig. 83 merely represents an instantaneous position of G, which is moving in a direction perpendicular to the diagram, and in accordance with the first law of motion, will continue so to move except in so far as it is compelled by forces to move in any other direction. There is no outward force exerted on the vehicle, and in any overturning motion, G will arrive outside the outer rail by a tangential rather than a radial path. Similarly, " side-slip " of a road vehicle on a curve represents a tangential continuation in a straight line of motion rather than radial motion.

speeds the resultant thrust on the track will fall to the left of C, with a greater load on A than on B, and F_A acting up the slope. And at the speed for which the banking is designed, the thrust will be through C, *i.e.* normal to the banking, and as in (1), $(Wv^2/gr) \div W = \tan \alpha$ or $v^2 = gr \tan \alpha$. For greater speeds, the horizontal centrifugal thrust of the vehicle will increase and the resultant thrust through G will be along some line such as GH. The reactions of the track at A and B together with the gravitational pull W produce on the vehicle an inward radial force Wv^2/gr to the left of G, or the reactions together are vectorially equal to W upward plus Wv^2/gr horizontally, leftward. Taking components of the reactions normal to the slope and tangentially down it, N_A and F_A at A, N_B and F_B at B, and resolving W and Wv^2/gr in these directions,

$$\text{Normally, } N_B + N_A = (Wv^2/gr) \sin \alpha + W \cos \alpha \quad . \quad . \quad (8)$$

$$\text{Laterally, } F_A + F_B = (Wv^2/gr) \cos \alpha - W \sin \alpha \quad . \quad . \quad (9)$$

And taking moments about G (about which there is no rotation) of the forces exerted on the vehicle,

$$(N_B - N_A)\tfrac{1}{2}d = (F_A + F_B)h$$
$$= Wv^2/gr h \cos \alpha - Wh \sin \alpha \quad (10)$$

$$N_B - N_A = W(2h/d)\{(v^2/gr) \cos \alpha - \sin \alpha\} \quad . \quad . \quad (11)$$

and from (8)

$$N_B + N_A = W\{(v^2/gr) \sin \alpha + \cos \alpha\}$$

By addition,

$$2N_B = Wv^2/gr\{(2h/d) \cos \alpha + \sin \alpha\}$$
$$+ W\{\cos \alpha - (2h/d) \sin \alpha\}$$

and by subtraction,

$$2N_A = Wv^2/gr\{\sin \alpha - (2h/d) \cos \alpha\}$$
$$+ W\{(2h/d) \sin \alpha + \cos \alpha\} \quad . \quad (12)$$

At the overturning speed, $N_A = 0$ and then from (12),

$$v^2 = gr(h \sin \alpha + \tfrac{1}{2}d \cos \alpha)/(h \cos \alpha - \tfrac{1}{2}d \sin \alpha)$$

or
$$v^2 = gr(h \tan \alpha + \tfrac{1}{2}d)/(h - \tfrac{1}{2}d \tan \alpha) \quad . \quad . \quad (13)$$

which reduces to $v^2 = grd/(2h)$ when $\alpha = 0$ as in (6).

The length CH is easily determined, say, by taking moments about C of the forces exerted by the vehicle on the track. $CH \times (N_B + N_A) = W \times EC - W(v^2/gr) \times EG$ where $EC = h \sin \alpha$ and $EG = h \cos \alpha$ or $HB/AB = N_A/(N_B + N_A)$; or taking GCH as a vector force diagram,

$$CH/h = (F_A + F_B)/(N_A + N_B).$$

On this slope, without rails and flanged wheels, side slip will occur when $F_A + F_B$ reaches a limit $\mu(N_A + N_B)$ and, using the values from (8) and (9), that occurs when

$$(Wv^2/gr) \cos \alpha - W \sin \alpha = \mu\{W \cos \alpha + (Wv^2/gr) \sin \alpha\}$$

or
$$v^2 = gr(\mu + \tan \alpha)/(1 - \mu \tan \alpha) \quad . \quad . \quad . \quad (14)$$

If we equate the right-hand sides of equations (13) and (14) we should have the conditions under which the vehicle was on the point both of side slip and of overturning, viz.,

$$h/d = 1/2\mu \text{ or } h = d/2\mu \text{ or } h = \tfrac{1}{2}d/\mu \quad . \quad . \quad (15)$$

which is independent of α and covers the case of the unbanked track ($\alpha = 0$).

It may be noted that on the steepest banking on which the (symmetrical) vehicle can remain stationary without sliding sideways to the inner side of the track, viz. when $\tan \alpha = \mu$, it is also, when stationary, on the point of overturning inwards if $h = \frac{1}{2}d/\mu$. And at the speed given by (13) or (14), it is on the point of side slipping and overturning outwards, for $\tan^{-1} CB/GC$ or $\tan^{-1} \frac{1}{2}d/h$ is the maximum inclination of the resultant thrust between the vehicle and track without overturning and without sliding if this angle is $\tan^{-1}\mu$. The speed for such banking is given by $v^2 = gr \times 2\mu/(1 - \mu^2)$, *i.e.* v is $\sqrt{2/(1 - \mu^2)}$ times that for side-slipping on an unbanked track as given by (7), say, 54 per cent. greater when $\mu = 0{\cdot}4$.

EXAMPLE 1. If on an unbanked track of 300 ft. radius a vehicle is to travel at 60 m.p.h., how high may the c.g. of the vehicle be without derailment if the distance between wheel centres is 5 ft. ?

From (6) $h = grd/2v^2$

$= 32{\cdot}2$ ft./sec.$^2 \times 300$ ft. $\times 5$ ft./(2×88^2)(ft./sec.)^2

$= 3{\cdot}12$ ft.

Note that to prevent side slip at this speed, the value of μ necessary would be $Wv^2/gr \div W = 88^2/32{\cdot}2 \times 300 = 0{\cdot}80$, a higher value than could be expected. Only a rail and flanged wheel could be expected to retain a vehicle on a track of such curvature at speeds approaching 60 m.p.h. (which would be much above a safe limit).

EXAMPLE 2. Find the height of the c.g. for overturning with the data of Example 1 but with track banked so that $\tan \alpha = 0{\cdot}1$.

Substituting in (13)

$$88^2 (\text{ft./sec.})^2 = 32{\cdot}2 \text{ ft./sec.}^2 \times 300 \text{ ft. } (0{\cdot}1h + 2{\cdot}5 \text{ ft.})/(h - 2{\cdot}5 \times 0{\cdot}1 \text{ ft.})$$

$$h(7744 - 966) \text{ ft.}^2 = (24{,}150 + 1936) \text{ ft.}^3$$

$$h = 26086/6778 = 3{\cdot}85 \text{ ft.}$$

(compare with 3·12 ft.

EXAMPLE 3. If a coefficient of friction of 0·2 can be relied upon, find at what speed a vehicle can get round a curve of 100 ft. radius without side slip (*a*) on a level road, (*b*) on a road banked to an angle of one vertically in eight horizontally. At what speed can the car travel on the banked road without any lateral frictional support?

(*a*) From (7), $v^2 = \mu gr = 0{\cdot}2 \times 32{\cdot}2 \text{ ft./sec.}^2 \times 100 \text{ ft.}$

$$v^2 = \sqrt{644} = 25{\cdot}4 \text{ ft./sec. or } 17{\cdot}3 \text{ m.p.h.}$$

(*b*) From (14),

$$v^2 = 32{\cdot}2 \text{ ft./sec.}^2 \times 100 \text{ ft. } (0{\cdot}2 + 0{\cdot}125)/(1 - 0{\cdot}2 \times 0{\cdot}125)$$

$$= 3220 \times 0{\cdot}325/0{\cdot}975 = 1098 \text{ (ft./sec.)}^2$$

$$v = 33{\cdot}14 \text{ ft./sec. or } 22{\cdot}6 \text{ m.p.h.}$$

With no lateral support, from (1)

$$v^2 = 0{\cdot}125 + 32{\cdot}2 \text{ ft./sec.}^2 \times 100 \text{ ft.} = 402{\cdot}5$$

$$v = 20{\cdot}1 \text{ ft./sec. or } 13{\cdot}64 \text{ m.p.h.}$$

EXAMPLE 4. If $\mu = 0{\cdot}4$, find the height of the c.g. in terms of the distance between the wheels, at which a symmetrical vehicle would be on the point of both side slipping and overturning on a track banked to a slope of one

vertically to five horizontally (*i.e.* tan $\alpha = 0{\cdot}2$). At what speed would a condition of overturning and side slipping be reached if the radius of curvature of the track is 100 ft.?

Substituting in (15), $h/d = 1/0{\cdot}8 = 1{\cdot}25$

And from (14)

$$v^2 = 32{\cdot}2 \text{ (ft./sec.}^2\text{)} \times 100 \text{ ft.} \times (0{\cdot}4 + 0{\cdot}2)/(1 - 0{\cdot}4 \times 0{\cdot}2)$$
$$= 3220 \text{ (ft./sec.)}^2 \times 0{\cdot}6/0{\cdot}92 = 2100 \text{ (ft./sec.)}^2$$
$$v = 45{\cdot}8 \text{ ft./sec. or } 31{\cdot}2 \text{ m.p.h.}$$

which may be checked by (13), putting $h/d = 1{\cdot}25$.

70. Dynamics of Simple Centrifugal Governor. Any general treatment of governors is outside the scope of this book, but very simple cases may be used to illustrate the importance of centrifugal force in the study of machinery.

Let Fig. 85 represent diagrammatically a simple centrifugal governor in which on one side an arm AB is

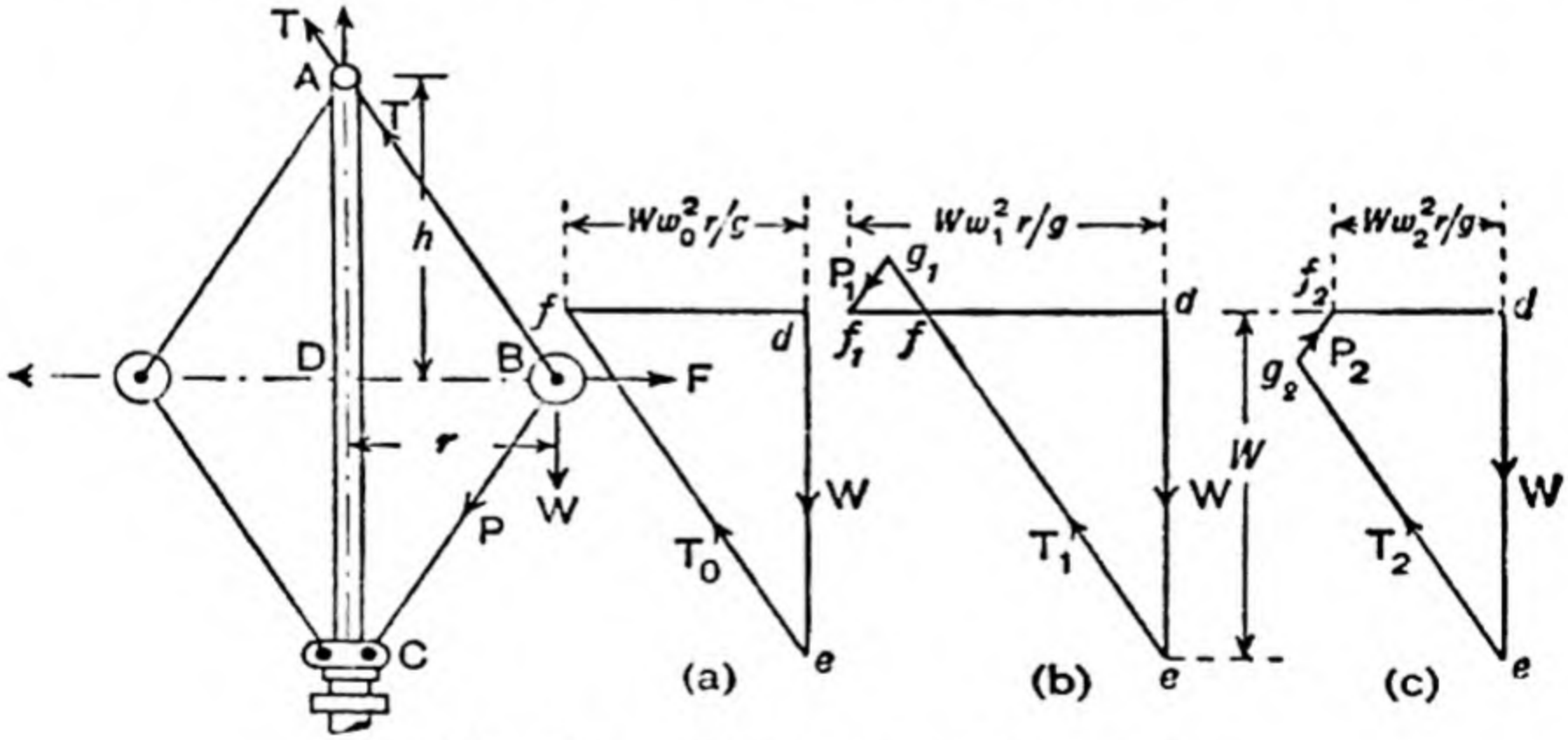

FIG. 85.—Forces on one arm of governor.

pin-jointed at its upper end A to a vertical rotating spindle and at its lower end to a ball B. A link BC is virtually an extension of the ball B (but of negligible weight) and is jointed at its lower end C to a sleeve which can slide on the revolving spindle and, by its movement, actuate a lever

and thereby operate a valve. The other side of the governor is similar. Consider the forces exerted on the arm AB when the governor is revolving with angular velocity ω_0. They are (1) the force exerted on the arm by the pin at A in the direction BA, (2) the weight W, (3) the centrifugal force F *exerted by the ball on the arm*, and (4) the force, if any, exerted by the link BC on the arm AB at B. If the arm AB is of negligible mass and therefore not subject to any (other than negligible) gravitational force arising from its own mass, nor requiring any but negligible force for acceleration of its parts in their circular paths about the central vertical spindle, the foregoing four forces will be in equilibrium. And we can consequently state their relations one to another by a closed vector polygon of forces or by equating opposing moments of these forces in the plane of the diagram, about some point in that plane. In the vector diagram (*a*), *de* is drawn vertically to represent W, the weight of the ball B, and *ef* is drawn parallel to BA to the level of *d*. Then if the horizontal line *fd* represents vectorially the centrifugal force F, of magnitude $W\omega_0^2 r/g$, the vector force diagram is a closed polygon (triangle) and there is no pull or thrust exerted by the link BC on the arm AB. (There will always be a small tension P_0 due to the weight of the link and sleeve.) Now suppose the angular velocity of the governor to increase from ω_0 to ω_1. If the link BC offers no resistance, the arm AB (constituting a conical pendulum) will take up a new position with reduced height, *i.e.* B will rise. But if, because of frictional or other resistance, the height does not change, a tension P_1 will be set up in the link BC. If in diagram (*b*) of Fig. 85, eg_1 is drawn parallel to BA and df_1 is drawn horizontally to represent $W\omega_1^2 r/g$, then a line from f_1 parallel to BC to meet eg_1 in g_1 will represent the pull exerted on the arm by the link BC in magnitude and the direction of the pull will be $g_1 f_1$. The closed vector polygon represents the vector sum $de + eg_1 + g_1 f_1 + f_1 d = 0$ or $W + T_1 + P_1 + W\omega_1^2 r/g = 0$. Finally, if instead of an increase of speed, there is a decrease to ω_2, by making $f_2 d$ in (*o*), Fig. 85, to represent $W\omega_2^2 r/g$ and completing the vector polygon $deg_2 f_2$ as before, we should get $g_2 f_2$ representing P_2, a force thrusting at B, *i.e.* there would be a compressive stress in the link BC, involving a thrust on the pin C, the

vertical component of which would be a force pushing the sleeve downwards. (With two balls this would be doubled.)

As a second illustration we take a slightly different type of governor, represented diagrammatically in Fig. 86, and

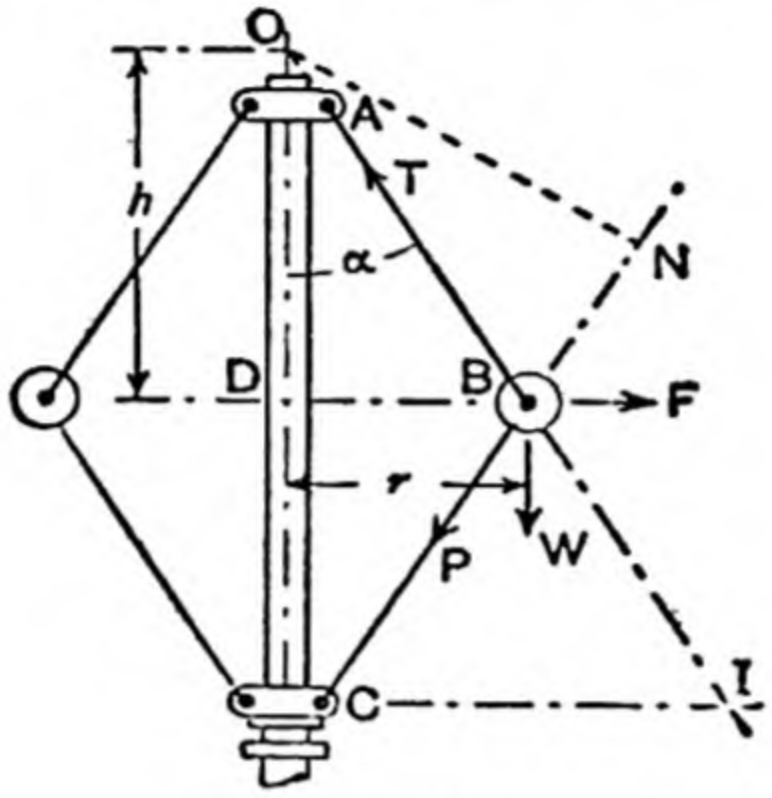

FIG. 86.

find the forces by equating opposite moments on the arm AB which has exerted upon it the forces W, F, T, and P as before. By taking moments about the point O at the intersection of AB with the vertical axis we eliminate any moment of T, the pull of the upper pin on the arm AB and we get

$$F \times h = W \times r + P \times ON \quad . \quad . \quad . \quad . \quad (1)$$

where P is the pull in the direction BC and ON is the perpendicular distance of the line BC from O.

If at speed ω_0, $F \,.\, h = W \,.\, r$, where $F = W\omega_0^2 r/g$ and $h = g/\omega_0^2$, $P = 0$ at speed ω_0. But, if the speed is increased to ω_1 without change in h and r the equation of moments about O becomes

$$h \times W\omega_1^2 r/g = Wr + P_1 \times ON \quad . \quad . \quad . \quad (2)$$

$$h \times W\omega_0^2 r/g = Wr \quad . \quad . \quad . \quad . \quad . \quad (3)$$

And since, by subtraction,

$$h \times W(\omega_1^2 - \omega_0^2) r/g = P_1 \times ON \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or

$$P_1 = Whr(\omega_1^2 - \omega_0^2)/(g \,.\, ON) \,. \quad (5)$$

Similarly, at the lower speed ω_2,

$$P_2 = Whr(\omega_2^2 - \omega_0^2)/g \,.\, ON \quad . \quad . \quad . \quad (6)$$

which is negative (*i.e.* a thrust) since ω_2 is less than ω_0.

The vertical component of P represents the vertical upward pull (or push) on the sleeve at C for one ball. It is evident that these values of P agree with those in Fig. 85, for $f_1f = Wr(\omega_1^2 - \omega_0^2)/g$, and the ratio h/ON is the same as f_1g_1/f_1f, hence $f_1g_1 = (h/ON) \times Wr(\omega_1^2 - \omega_0^2)/g$, as found for P_1.

Since from (3), $h = g/\omega_0^2$ as for the simple conical pendulum, P_1 from (5) may be written

$$P_1 = Wr(\omega_1^2 - \omega_0^2)/(\omega_0^2 \,.\, ON) \quad . \quad . \quad . \quad (7)$$

and the *ratio* $(\omega_1^2 - \omega_0^2)/\omega_0^2$ will not be altered if we write revolutions per minute instead of the corresponding number of radians per second, thus

$$P_1 = Wr(N_1^2 - N_0^2)/N_0^2 \times ON \quad . \quad . \quad (8)$$

and similarly for (6)

$$P_2 = Wr(N_2^2 - N_0^2)/N_0^2 \times ON \quad . \quad . \quad (9)$$

If both links and arms are inclined at an angle α to the vertical, $r = h \tan \alpha$, and $ON = 2h \sin \alpha$, and from (7)

$$P_1 = \tfrac{1}{2}W\{(\omega_1^2/\omega_0^2) - 1\}\sec \alpha \quad . \quad . \quad . \quad (10)$$

and the vertical pull on the sleeve (per ball) is

$$W' = P_1 \cos \alpha = \tfrac{1}{2}W\{(\omega_1^2/\omega_0^2) - 1\}$$
$$= \tfrac{1}{2}W\{(N_1^2/N_0)^2 - 1\} \quad . \quad (11)$$

If the inclinations of arms and link are not equal one to the other, $P_1 \cos \alpha$ may be found from (7) or a formula may be found to express ON.

The vertical pull W′ is independent of the value of α as shown in (11). This might have been found by resolving P_1 at B into a vertical component W′ and a horizontal component W′ tan $\alpha = W' \times r/h$. The moment $P \times ON$ about O from the moments of its components is

$$P \times ON = W' \times r + W' \times (r/h) \times h = 2W'r$$

hence from (7)

$$2W'r = Wr(\omega_1^2 - \omega_0^2)/\omega_0^2$$

or

$$W' = \tfrac{1}{2}W\{(\omega_1^2/\omega_0^2) - 1\} \text{ as in (11).}$$

Limitations of Simple or Unloaded Governor. The height of the conical pendulum or unloaded governor is $h=g/\omega^2$, and if g is in ft./sec.^{2}, h will be in ft. Thus at 60 r.p.m., $\omega=2\pi$ radn./sec. and h is about 0·8 ft. or, say, 9·6 inches. With increase of speed, h will become impracticably small, *e.g.* at 180 r.p.m. it will be $(1/3)^2$ or 1/9 of 9·6 inches. A reasonable height h and size of governor can be used by increasing the controlling force (which counterbalances the centrifugal force) by gravitational force or by springs. By reference to equation (2) it is evident that at the higher speed ω_1 the values of h and r appropriate to the speed ω_0 (viz. $h=g/\omega_0^2$) may be maintained by providing a sufficient pull P in the link BC. This is done in the loaded type of governor by adding a considerable weight to the sliding sleeve. The clockwise moment about O then exerted by P is adequate to balance a higher centrifugal force F corresponding to a higher speed ω_1.

Application of Principle of Work to Governors. The relations of the forces, speed, and dimensions in a governor may conveniently be established by the Principle of Work (Art. 46). This may be briefly illustrated by the loaded governor. Suppose a gravitational load W′ *per ball* to be on the sleeve of the governor in Fig. 86. The forces exerted on the weightless link BC (exclusive of the ball) are in equilibrium and are (1) the centrifugal force F, (2) W′ vertically at C, and (3) the pull T at B. The instantaneous centre of motion of BC is at I, the intersection of AB produced with a horizontal line through C. In the general case we might well draw the vector velocity diagram, but with the simple case of links and arms all of equal length, we can write $IC=2r$, $IB=h\sec\alpha$, where α is the angle of the arms and links to the vertical.

Omitting the common factor of the angular velocity of all points of BC about I, and equating the *rate* of work done by F to that done on W and W′

$$F\times IB\cos\alpha=W\times IB\sin\alpha+W'\times IC$$

$$(W\omega^2r/g)\times h=W\times r+W'\times 2r$$

$$h=(g/\omega^2)(1+2W'/W) \quad . \quad . \quad . \quad (12)$$

and

$$\omega^2=(g/h)(1+2W'/W) \quad . \quad . \quad . \quad (13)$$

Comparison of (12) with the height g/ω^2 for the unloaded governor ($W'=0$) shows that the height of the loaded one is greater in the ratio $(1+2W'/W)$ to one.

EXAMPLE 1. The spindle of a simple centrifugal governor runs at 50 r.p.m. without friction and the two balls each weigh 12 lb. Find the vertical upward pull on the sleeve if the speed increases by 10 per cent. without change in the height of the balls, if the arms and links are equally inclined to the vertical axis. Without friction, what load on the sleeve would be required for the governor to run at 125 r.p.m. ?

From (11), for 10 per cent. increase in speed

$$W=\tfrac{1}{2}\times 12\times(1\cdot1^2-1)=6\times 0\cdot21=1\cdot26 \text{ lb. per ball.}$$

Total vertical pull $=2\times 1\cdot26=2\cdot52$ lb.

For a speed of 125 r.p.m., $N_1/N_0=125/50=2\cdot5$

$$W'=\tfrac{1}{2}\times 12\times(2\cdot5^2-1)=6\times 5\cdot25=31\cdot5 \text{ lb. per ball.}$$

Total vertical pull to be balanced by a load

$$=2\times 31\cdot5 \text{ lb.}=63 \text{ lb.}$$

Note that the height h for 50 r.p.m. (unloaded) would be

$$h=g/\omega^2=32\cdot2/(100\pi/60)^2=32\cdot2/27\cdot41=1\cdot1747 \text{ ft., and}$$

for 125 r.p.m. unloaded, h would be $1\cdot1747/2\cdot5^2$ or 0·188 ft., which is a quite impracticably small height and an unloaded governor could not be used.

EXAMPLE 2. Neglecting friction, find the height of a simple governor running at 50 r.p.m. and the speeds when the balls (*a*) rise one inch, (*b*) fall one inch.

As in Example 1, for 50 r.p.m., $h=1\cdot1747$ ft.

When the balls rise 1 inch or 0·0833 ft., $h=1\cdot0914$ ft.

$$\omega_1/\omega_0=N_1/50=\sqrt{1\cdot1747/1\cdot0914}=\sqrt{1\cdot0993}=1\cdot0496$$

$$N_1=50\times 1\cdot0496=52\cdot48 \text{ r.p.m.}$$

When the balls fall 1 inch, $h=1\cdot258$ ft.

$$\omega_2/\omega_0=N_2/50=\sqrt{1\cdot1747/1\cdot254}=\sqrt{0\cdot9338}=0\cdot9663$$

$$N_2=50\times 0\cdot9663=48\cdot3 \text{ r.p.m.}$$

EXAMPLE 3. If the governor in Example 1 has a frictional resistance which may amount to 2 lb. in either direction at the sleeve, find the speed limits without movement of the governor (*a*) without load, (*b*) with the 63 lb. sleeve load.

The total resistance of 2 lb. is 1 lb. per side, so that considering one side only, W′ for condition (*a*) is ± 1 lb. and for (*b*) is $(31{\cdot}5 \pm 1)$ lb.

(*a*) For the higher speed limit ω_1, say, by (13)

$$\omega_1^2 = (g/h)(1 + 2/12) = (g/h)(1 + 0{\cdot}1667)$$

$$\omega_0^2 = g/h$$

$$\omega_1^2/\omega_0^2 = N_1^2/50^2 = 1{\cdot}1667$$

$$N_1/50 = \sqrt{1{\cdot}1667} = 1{\cdot}080 \quad N = 54{\cdot}0 \text{ r.p.m.}$$

For the lower limit ω_2, say,

$$\omega_2^2/\omega_0^2 = N_2^2/50^2 = 1 - 0{\cdot}1667 = 0{\cdot}8333$$

$$N_2/50 = \sqrt{0{\cdot}8333} = 0{\cdot}9129, \; N_2 = 45{\cdot}6 \text{ r.p.m.}$$

The range of speed is $54{\cdot}0 - 45{\cdot}6 = 8{\cdot}4$ r.p.m. or 16·8 per cent. of 50 r.p.m.

(*b*)

$$\omega_1^2 = (g/h)\{1 + (2 \times 32{\cdot}5)/12\} = (g/h)(6{\cdot}4167)$$

$$\omega_0^2 = (g/h)(1 + 5{\cdot}25) = (g/h)(6{\cdot}25)$$

$$N_1^2/125^2 = 6{\cdot}4167/6{\cdot}25 = 1{\cdot}02667$$

$$N = 125 \times \sqrt{1{\cdot}02667} = 125 \times 1{\cdot}0133 = 126{\cdot}7 \text{ r.p.m.}$$

$$N_2^2/125^2 = \omega_2^2/\omega_0^2 = \{1 + 2 \times 30{\cdot}5/12\}/(1 + 5{\cdot}25)$$

$$W_2 = 125 \times \sqrt{6{\cdot}0833/6{\cdot}25} = 125 \times 0{\cdot}9866$$

$$= 123{\cdot}3 \text{ r.p.m.}$$

The range of speed is $126{\cdot}7 - 123{\cdot}3 = 3{\cdot}4$ r.p.m. or 2·7 per cent. of 125 r.p.m.

71. Balancing. In one Plane of Revolution. Any mass attached to a shaft and revolving with it about the axis of the shaft will exert a pull on the shaft, viz. the centrifugal force

which the mass exerts on the connection which constrains it to move in a circle. This pull is perpendicular to the axis of the shaft and tends to bend it, and moreover it causes a pull on the bearing of the shaft. Such forces on the bearings in any given direction are periodic, reaching equal and opposite maxima once per revolution and so tend to set up vibrations in whatever structure holds the bearings. If a second mass is attached to the shaft with its c.g. in the same plane of rotation as that of the first one and diametrically opposite to it and exerting an equal centrifugal in the same straight line as the first one, but in the opposite direction, the second mass will *balance* the first one and no resultant centrifugal force will be exerted on the shaft by the two masses.

Let W_1, Fig. 87, be the first mass and W_2 be the second one, and let r_1 and r_2 be the respective distances of their

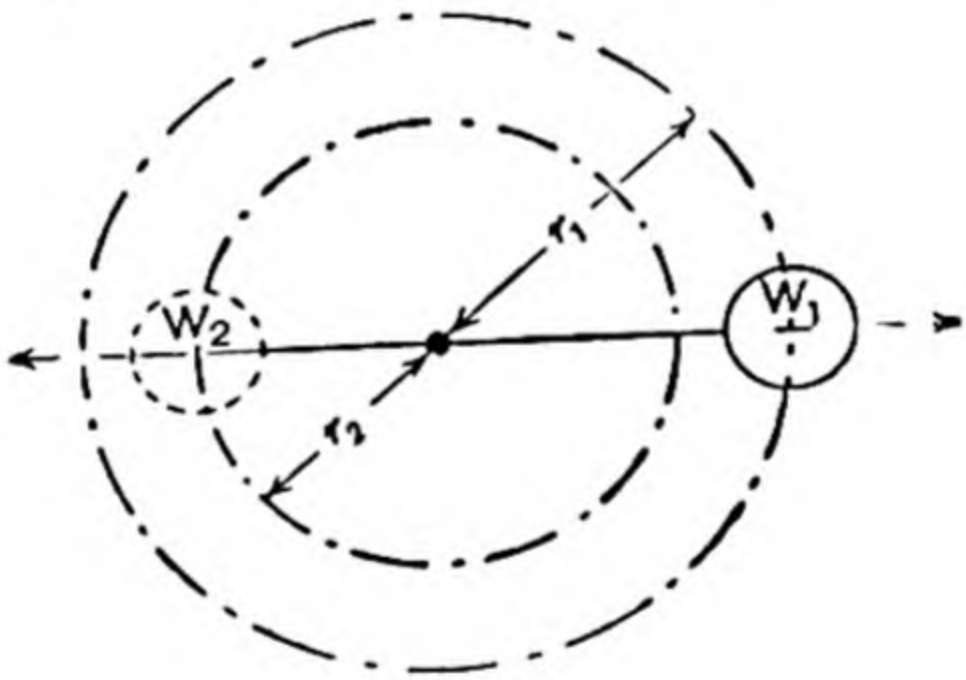

FIG. 87.

centres of gravity from the axis of the shaft. Then in order that their centrifugal pulls on the shaft shall balance one another,

$$W_1\omega^2 r_1/g = W_2\omega^2 r_2/g \quad . \quad . \quad . \quad . \quad (1)$$

where ω is the angular velocity of the shaft and attached masses. Evidently this is satisfied if

$$W_1 r_1 = W_2 r_2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the factor ω^2/g may be omitted in such equations of centrifugal forces exerted on the shaft. Note that the

c.g. of the combined masses W_1 and W_2 is then in the axis of the shaft.

But a mass with its c.g. diametrically opposite to that of W_1 is not the only means of arranging a centrifugal pull within this plane of rotation, equal and opposite to that exerted by W_1. Two masses arranged so that the resultant of the centrifugal forces which they exert on the shaft is equal and opposite to that of W_1 will balance it. In other words, if the vector sum of the centrifugal forces of two balance weights W_2 and W_3, say, is equal and opposite to the centrifugal force of W_1, or if the vector sum of the centrifugal forces of W_1, W_2, and W_3 is zero, the centrifugal forces (or the masses) are balanced. This is illustrated in Fig. 88, *abc* being the closed vector triangle of forces. And obviously we are not limited to two balance weights;

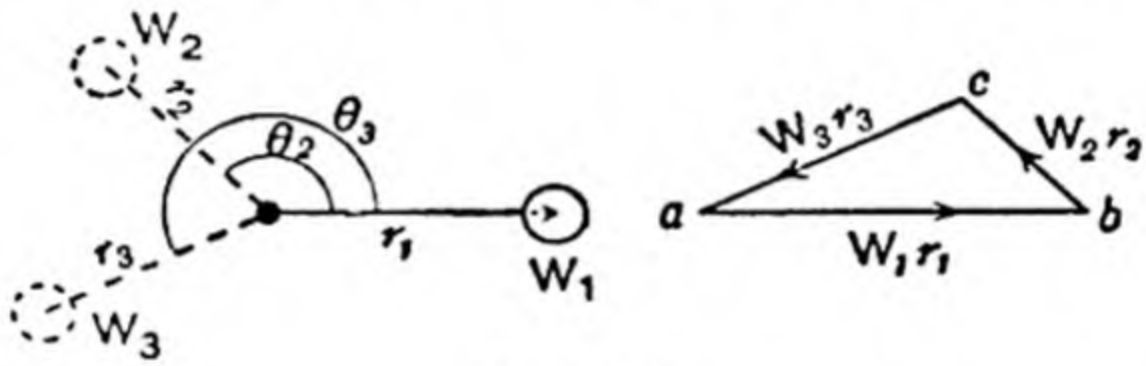

FIG. 88.

and it is true generally for several masses having their centres of gravity all in one plane of rotation, that they are in balance among themselves if the vector force polygon of sides proportional to the products Wr and directions parallel to the radii r, is a closed figure. The condition of balance is the same as that for any set of concurrent forces in one plane (Art. 20), and may be expressed by the closing of the vector force polygon or by the equations (7) of Art. 21,

$$H=0,\quad V=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where H and V are two sets of rectangular components of the centrifugal forces $\Sigma(P\cos\theta)$ and $\Sigma(P\sin\theta)$, or

$$\Sigma(WR\cos\theta)=0,\quad \Sigma(WR\sin\theta)=0 \quad . \quad . \quad (4)$$

It may be noticed that in case of balance of masses in one plane, the c.g. of all the revolving masses (combined) is in the axis of revolution. If any revolving mass has its

c.g. not in the axis, it exerts a resultant centrifugal force. If it (or a group of masses) has its c.g. in the axis, it exerts no resultant centrifugal force on whatever constrains it to revolve in a circular path. (There will be stresses set up in the material of the mass and its constraints.)

EXAMPLE 1. A mass A of 20 lb. with its c.g. 6 inches from the axis of rotation is to be balanced by masses B and C, each of 10 lb. If the angular distances of the centres of gravity of B and C from that of A are 135° and 210° respectively, measured in the same direction from A, find the radii at which the c.g. of B and C should be placed.

Using equation (4), $\Sigma(Wr\cos\theta)=0$, if r_B and r_C are the radii to c.g. of B and C respectively,

$$20 \text{ lb.} \times 6 \text{ in.} + Br_B \cos 135° + Cr_C \cos 210° = 0$$

$$0{\cdot}7071\ Br_B + 0{\cdot}866\ Cr_C = 120 \text{ lb.-in.}$$

$$\Sigma(Wr \sin\theta)=0$$

and using

$$120 \sin 0° + Br_B \sin 135° + Cr_C \sin 210° = 0$$

$$0 + 0{\cdot}7071 Br_B - 0{\cdot}5\ Cr_C = 0$$

Subtracting this from the first equation

$$1{\cdot}366\ Cr_C = 120 \text{ lb.-in.}$$

$$Cr_C = 120/1{\cdot}366 = 87{\cdot}86 \text{ lb.-in.}$$

$$Br_B = 0{\cdot}5 \times 87{\cdot}86/0{\cdot}7071 = 62{\cdot}12 \text{ lb.-in.}$$

And since B and C are each 10 lb.,

$$r_B = 62{\cdot}12/10 = 6{\cdot}21 \text{ in.}, \quad r_C = 87{\cdot}86/10 = 8{\cdot}79 \text{ in.}$$

Alternatively, or as a check, from a vector triangle such as *abc* in Fig. 88, if $ab = 120$ lb.-in., $bc = 62$ lb.-in., $ca = 88$ lb.-in.

EXAMPLE 2. Three masses of 15, 10, and 6 lb. have their centres of gravity all revolving in one plane about an axis from which their distances are 4, 5, and 8 inches respectively, the second and third being in angular positions 90° and 135°, say clockwise, from the first. Find the magnitude of a single mass to balance this, centred 12 inches from the axis, and specify its angular position with respect to the first mass.

Let θ be the angle clockwise. Then from (4),

$$15 \text{ lb.} \times 4 \text{ in.} \times \cos 0° + 10 \text{ lb.} \times 5 \text{ in.} \times \cos 90° + 6 \text{ lb.} \times 8 \text{ in.} \times \cos 35° + 12 \text{ in. W} \cos\theta = 0$$

$$60 + 0 - 48 \times 0{\cdot}7071 + 12\text{W} \cos\theta = 0$$

$$12\text{W} \cos\theta \text{ (in.)} = -26{\cdot}06 \text{ lb.-in.}$$

Also

$$60 \sin 0° + 10 \text{ lb.} \times 5 \text{ in.} \times \sin 90° + 6 \text{ lb.} \times 8 \text{ in.} \times \sin 135° + 12 \text{ W} \sin\theta = 0$$

$$12\text{W} \sin\theta \text{ (in.)} = -83{\cdot}9 \text{ lb.-in.}$$

Dividing by the former equation

$$\tan\theta = 83{\cdot}9/26{\cdot}06 = 3{\cdot}221 = \tan 252{\cdot}75°$$

the acute angle 72·75° not satisfying the requirement that sin θ and cos θ must be negative.

$\text{W} = -83{\cdot}9/12 \times (-0{\cdot}955) = 7{\cdot}32$ lb. This and the angle $\theta = 252{\cdot}75°$ can easily be verified by a vector polygon of sides Wr, etc.

72. Balancing. In more than one Plane of Revolution. (*a*) *One Weight.* It is not always convenient, or even possible, to balance a mass by other masses with their centres of gravity in the plane of revolution of the first one. But a force can be balanced by two or more forces not concurrent with it provided they are parallel to it. Thus the centrifugal force of a mass W_1, Fig. 89, can be balanced by those of two other masses, W_2 and W_3, *in the same axial plane*, provided they have a sum equal to the centrifugal force of W_1 (proportional to W_1r_1) and so spaced along the axis as to produce, with W_1r_1, no resultant couple in the axial plane of the centres of gravity of the masses (which is the plane of the diagram). The conditions are, of course, those of elementary statics for the equilibrant of two or more parallel forces, and if the moment of the forces is zero about any one point in the plane, it is zero for all points in the plane. We may conveniently equate the moments about the point A for which the moment of the centrifugal force (W_1r_1) of W_1 vanishes. The conditions of balance are

$$W_1r_1 = W_2r_2 + W_3r_3 \quad . \quad . \quad . \quad . \quad (1)$$

$$W_2r_2 \times d_2 = W_3r_3 \times d_3 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We may notice that if W_2r_2 were equal to W_1r_1 and W_3 were zero, the magnitude of the resulting unbalanced couple rocking the shaft (and causing reactive forces on its bearings) would be $W_1r_1 \times d_2 \times \omega^2/g$ or $W_2r_2d_2\omega^2/g$. Also that even if conditions (1) and (2) are satisfied, while there is no moment or couple on the shaft as a whole required to balance the moment of other external forces such as reactions at bearings, there is a moment to be resisted by the shaft and which causes bending. For example, at A the shaft is subject to a bending moment ω^2/g times $W_2r_2d_2$ or $W_3r_3d_3$ and to lesser amounts at other points between B and C.

We may also note that if W_3 were to be balanced by

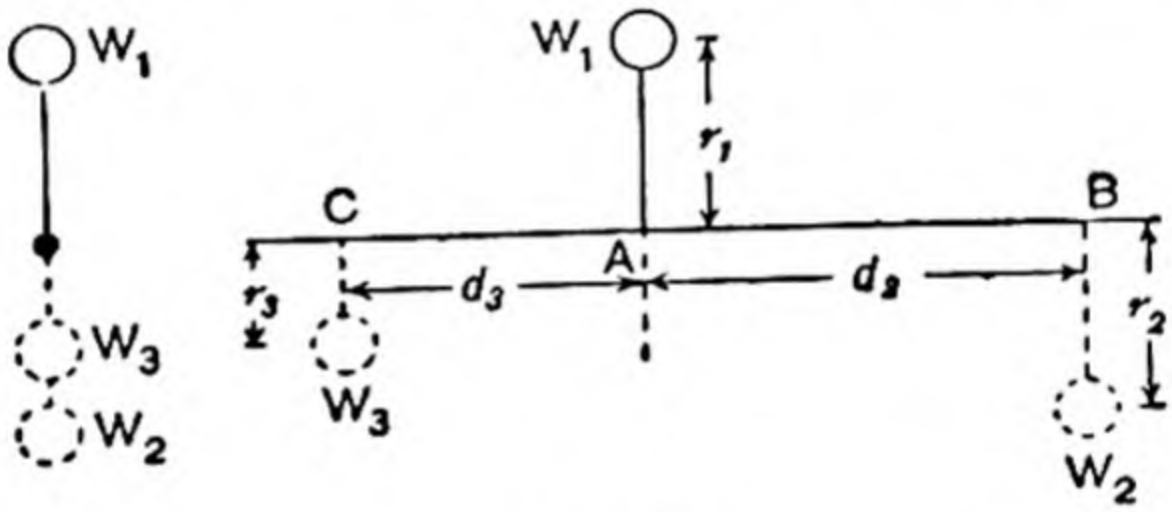

FIG. 89.

W_1 and W_2, both to the same side of W_3 (as in fact it is by the conditions (1) and (2)), the balance masses W_1 and W_2 must lie on opposite sides of the shaft axis BC in accordance with the elementary statics of parallel forces, viz. in order to balance a force by two parallel forces on the same side of it these two forces must be in opposite directions one to the other.

Either or each of the masses W_2 and W_3 may be replaced (in accordance with the principles of Art. 71 illustrated in Fig. 88) by two masses in the same plane of revolution (normal to the axis) by arranging that the vector sum of the centrifugal forces of the replacing masses shall be equal to the centrifugal force of the mass replaced.

A little consideration will show that this method of balancing a single mass by two others in different planes of revolution may be extended to balancing more than one mass of several, not all in the same axial plane. For if

we balance each mass by two masses in the (normal) planes of revolution through B and C, we should then have a series of provisional balance weights in those planes, and such balance weights can be replaced in accordance with the method employed in Art. 71, Fig. 88, by single masses in each of these two planes of revolution. The single mass would be placed at such a position that its centrifugal force would be the resultant of the centrifugal forces of the several (provisional) balance weights. This will be the case if its force vector is equal to the vector sum of the centrifugal forces of the several provisional balance masses found for the same plane of revolution. We proceed to illustrate this by finding the balance weights in given planes of revolution for two unbalanced masses not in the same axial plane nor in the same (normal) plane of revolution.

(*b*) *Two Weights.* The two masses W_1 and W_2 of Fig. 90 are to be balanced by two masses in the planes of rotation B and C. To determine them we first find the (provisional) centrifugal forces in B and C, opposite to W_1, but in the same axial plane as its centre of gravity, which will balance the centrifugal force W_1r_1 (omitting the factor ω^2/g). Secondly, we find the centrifugal forces in B and C which will balance the force W_2r_2 which is not in the same axial plane as W_1r_1, and finally we find the resultant of the two centrifugal forces in the plane B and the resultant of the two in plane C. In Fig. 90, W_1 has its c.g. in the axial plane of the diagram (*a*), but this is not the case for W_2 and the position of W_2 can be seen in the end views (*b*) and (*c*). These end views are in British Standard or first angle projection, but show only the balance forces in B at (*b*) only, and those of C at (*c*). The provisional or component centrifugal forces to balance W_1r_1 are

$$\text{In plane B, } w_3r_3 = W_1r_1 \times EA/EF$$

$$\text{In plane C, } w_5r_5 = W_1r_1 \times FA/EF$$

And the provisional centrifugal forces to balance W_2r_2 are

$$\text{In plane B, } w_4r_4 = W_2r_2 \times ED/EF$$

$$\text{In plane C, } w_6r_6 = W_2r_2 \times FD/EF$$

These centrifugal forces are shown by vectors at (*b*) and (*c*)

of Fig. 90. Then in plane B at (*b*), the vector sum of w_3r_3 and w_4r_4 is shown as w_7r_7, while in plane C at (*c*) the vector sum of w_5r_5 and w_6r_6 is shown as w_8r_8. And suitable balance masses in the radial directions of those vectors will balance W_1 and W_2 completely. The balance masses are indicated as W_B in plane B and W_C in plane C at the outer ends of the vectors so that they can be projected clearly (as dotted circles) in diagram (*a*), but the actual length of radius depends upon the magnitude of the mass adopted.

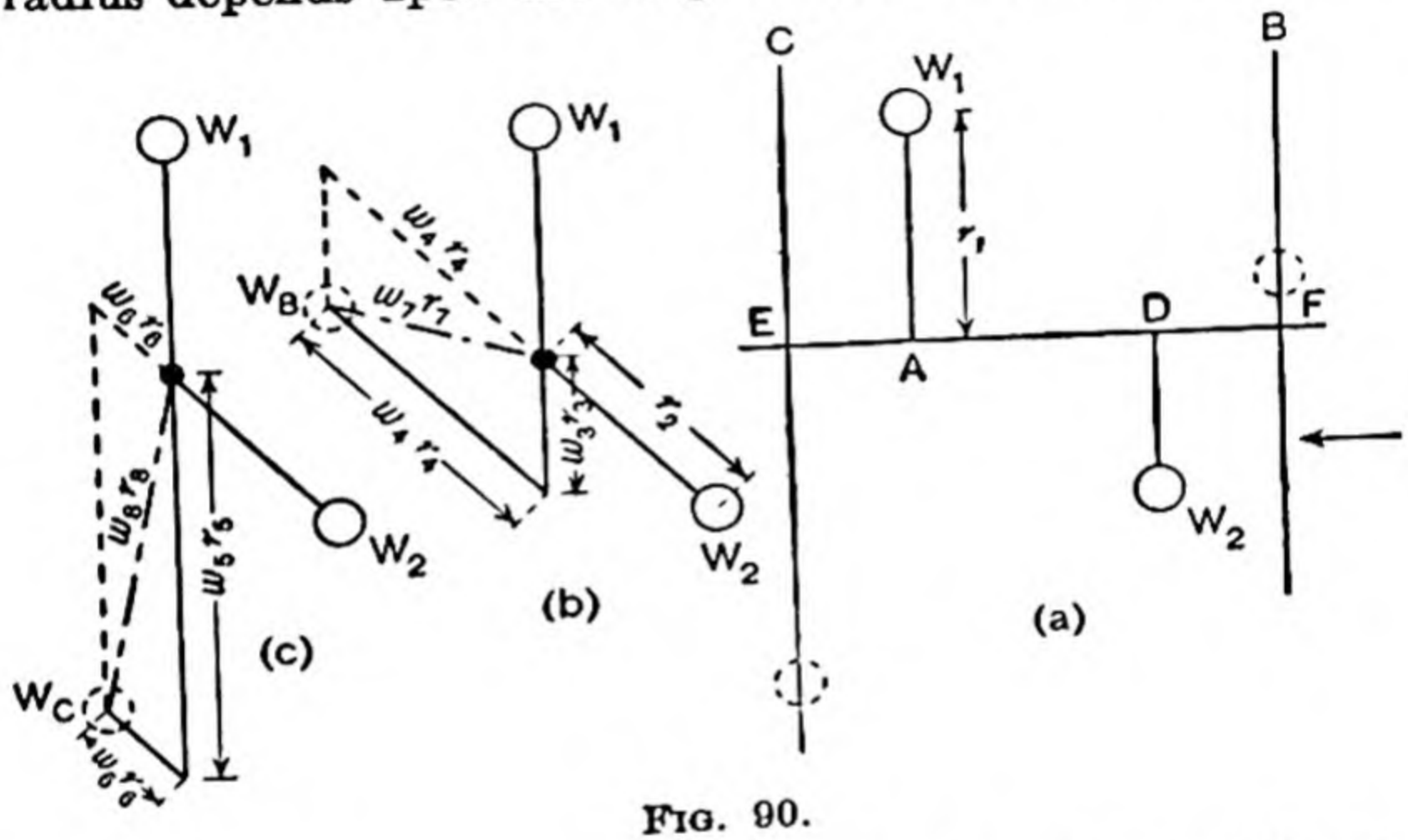

FIG. 90.

The radius of the c.g. of W_B is w_7r_7/W_B and the radius to the c.g. of W_C is w_8r_8/W_C.

This process of finding the balance weights in two planes of rotation (B and C) is equally applicable to more than two unbalanced masses, but in place of the triangles (or parallelograms) for the resultants of the pairs of forces $w_3r_3+w_4r_4$ and $w_5r_5+w_6r_6$, we should have open polygons of more than two sides.[1] While the graphical or vector

[1] Since all the centrifugal forces exerted on the shaft are in equilibrium, their projections (or components) on any axial plane are in equilibrium, and therefore their moments about any point in that plane are equal and opposite or balanced, having a zero sum. These might be stated in respect to two mutually perpendicular axial planes. But the same relation is covered by saying that a vector polygon having sides representing the

polygon method has been used for explanation, combination of the several provisional centrifugal balancing forces may of course be made by resolution into rectangular components, algebraic addition and recombination in a single resultant.

Standing and Running Balance. If several masses attached to and rotating with a shaft are such that their centrifugal forces (proportional to W*r*) have zero resultant, (*i.e.* if equation (1) is satisfied), the shaft is said to have a *standing* or *static* balance. If placed freely between horizontal axial pivots at its ends, the shaft and attached masses would stand in any angular position of the shaft. For the moments of the gravitational forces of the masses about the shaft axis would have zero sum, being the projected values on a horizontal plane of the centrifugal force products, *i.e* proportional to W*r*, etc. The standing balance would still hold even though the centrifugal forces might have a resultant moment about a point in the axis, or, rather, about an axis perpendicular to that of the shaft. If, in addition, the centrifugal forces have no resultant moment about any axis perpendicular to that of the shaft (*i.e.* if equation (2) is satisfied), the masses are said to have also a *running* or *dynamic* balance.

EXAMPLE. A locomotive has two cylinders with centres 2·5 ft. apart and cranks at right angles one to another. The balance weights on the wheels will have their centres of gravity 5 ft. apart and will be at a radius of 2 ft. from the axis of the crankshaft. Find the masses required and

product of the several centrifugal forces and their respective distances in an axial direction from a point in the axis forms a closed polygon. The inclinations of the sides are those of the various centrifugal forces, *i.e.* of the radii in the planes of revolution, joining the centres of gravity of the masses to the axis, and the polygon is in a plane perpendicular to the axis. This forms the basis of calculation known as Dalby's method and the axial distances by which the centrifugal force factors, W*r*, are multiplied, are measured from any " plane of reference " chosen for greatest convenience. Unless it is chosen so that all the centrifugal forces are on the same side of it, those on opposite sides will require opposite signs; the directions of the vectors of, say, negative products will be reversed in drawing the vector polygon of products (W.*r*.*d*, etc.). The development of this method, though simple, is outside the scope of this volume.

their positions if the masses to be balanced are 600 lb. at each crank pin and the crank throw is 1 ft.

The arrangement is shown in Fig. 91 where w_3 and w_6 are the provisional or component masses to balance W_1, w_4 and w_6 those to balance W_2. Taking moments about a vertical axis through the centre line AD, say an axis AH,

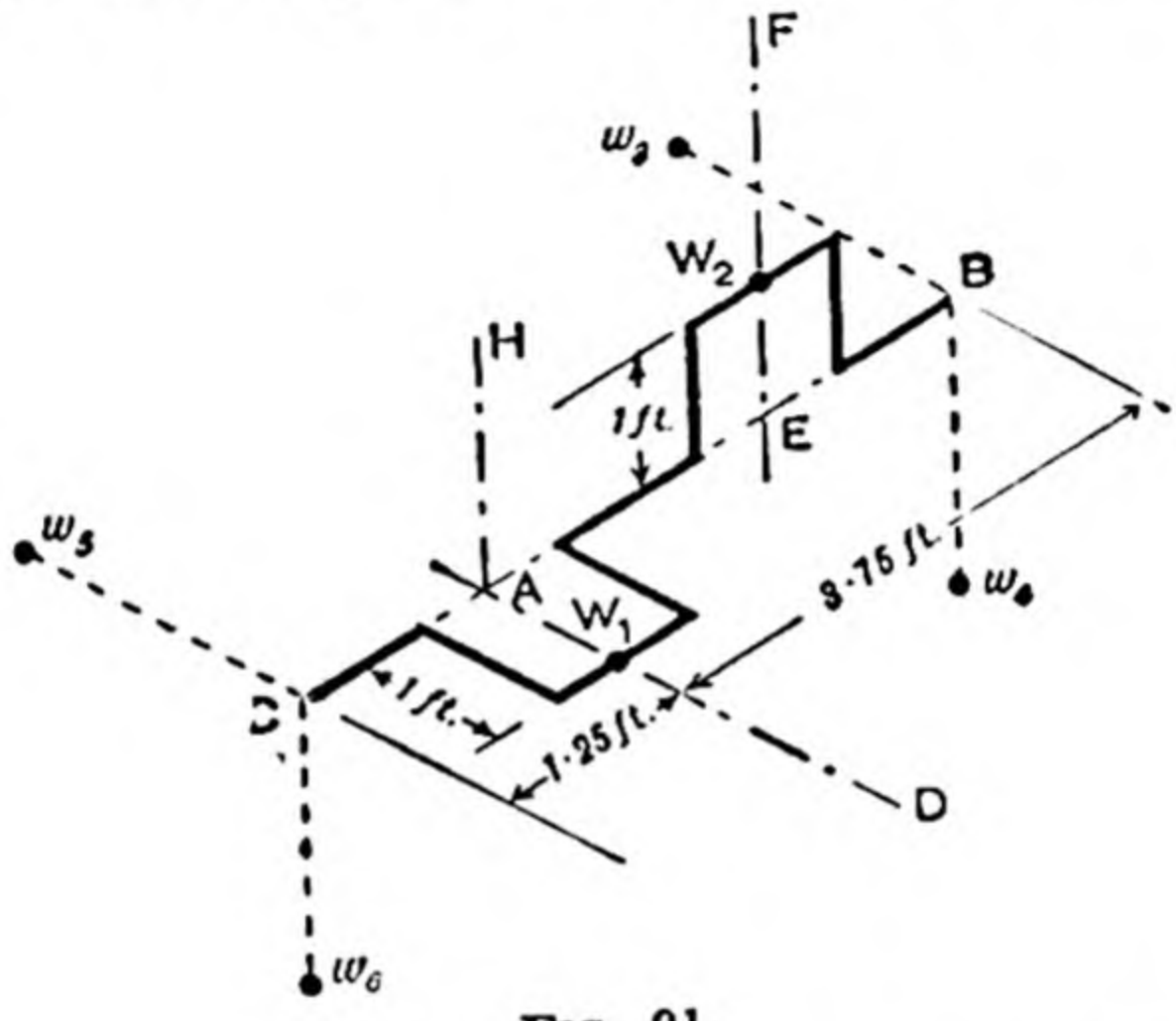

FIG. 91.

of the horizontal forces in the horizontal plane CABD (and omitting the common factor ω^2/g),

$$w_6 \times 2 \text{ ft.} \times 1{\cdot}25 \text{ ft.} = w_3 \times 2 \text{ ft.} \times 3{\cdot}75 \text{ ft.}$$

$$w_6 = 3w_3$$

And equating opposite horizontal forces,

$$2 \text{ ft.} \times (w_6 + w_3) = 600 \text{ lb.} \times 1 \text{ ft.}$$

$$w_6 + w_3 = 300 \text{ lb.}$$

and since $w_6 = 3w_3$,

$$4w_3 = 300 \text{ lb.}$$

$$w_3 = 75 \text{ lb.}$$

$$w_6 = 225 \text{ lb.}$$

And from the moments about a horizontal axis through

FE, w_4 and w_6 may be found, but by the symmetry obviously $w_4 = w_5 = 225$ lb., and $w_6 = w_3 = 75$ lb.

The resultant centrifugal force due to w_5 and w_6 is proportional to 2 ft. $\times \sqrt{225^2 + 75^2}$ and the mass required at 2 ft. radius on wheel C is

$$\sqrt{225^2 + 75^2} = 75\sqrt{9+1} = 237{\cdot}2 \text{ lb.}$$

And its c.g. radius vector makes an angle θ with the radius vector of w_5 such that $\tan\theta = 225/75 = 3$, *i.e.* $\theta = 71{\cdot}5°$, or at an angle of $180 + 71{\cdot}5 = 251{\cdot}5°$ from the near crank measured in the angular direction of the far crank. Correspondingly the other balance weight of 237·2 lb. on wheel B is to be placed 251·5° from the crank nearer to B measured in the angular direction of the crank more remote from B. It will be noticed that in this example, calculation is simplified by the fact that the centrifugal forces of W_1 and W_2 and their provisional equilibrants form two independent sets of forces which, being in mutually perpendicular planes, have components only in their own planes. In the more general case, either a vector polygon will be required or the forces will need to be resolved into two components in two mutually perpendicular planes.

(Checking by moments about a horizontal axis through C perpendicular to CB, 600 lb. $\times$1 ft. $\times$3·75 ft. $+$237·2 lb. $\times$2 ft. $\times$5 ft. $\times$ sin 251·5° $=$ (2250 $-$ 2250) lb.-ft.$^2 = 0$.)

73. Centrifugal Stress. When a body rotates, any part of it will be constrained to follow a circular path by the adjoining material, which will exert upon it an inward centripetal force, and will itself be subjected to a centrifugal force. Thus the dual forces across an imaginary boundary in the material will constitute a stress (Art. 93). We only need consider the case of a thin circular ring or rim of material rotating about its central axis, as shown in Fig. 92, where r is the radius of the ring, v its linear speed and $\omega = v/r$ is its angular velocity. The results will also apply to the centrifugal tension in a belt running on a pulley.

If we consider the upper half of the ring, its c.g. at G is at a radius $2r/\pi$ from O and it will therefore have an inward radial force

$$W\omega^2 \times (2r/\pi)/g \quad . \quad . \quad . \quad . \quad . \quad (1)$$

exerted upon it, where W is its weight, to keep it rotating about O. If w is the weight of the material per unit volume and A is its cross-sectional area

$$W = \pi r \times A \times w \quad . \quad . \quad . \quad . \quad . \quad (2)$$

And the inward radial force (1) is

$$\pi r \times A \times w \times \omega^2 \times 2r/(\pi g) = 2A\omega^2 r^2 w/g \quad . \quad . \quad (3)$$

This is transmitted from the other half of the ring by

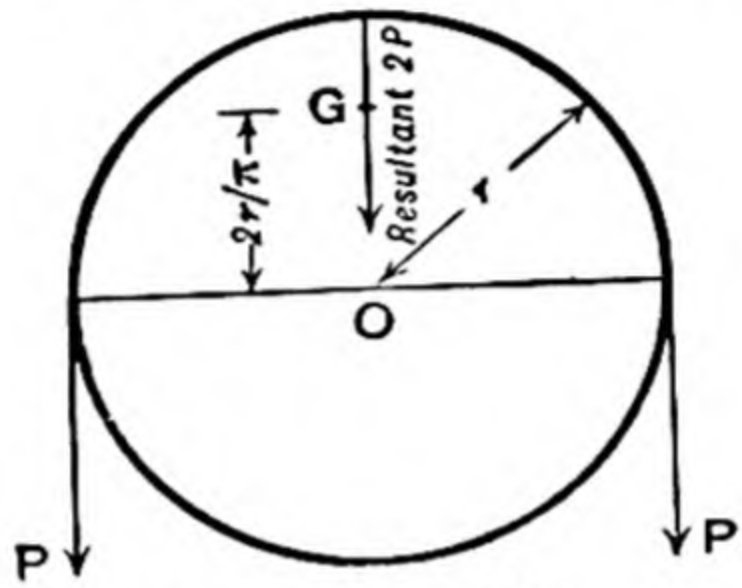

FIG. 92.—Centrifugal tension.

the total tension P at each diametrical boundary between the two halves, hence

$$2P = 2A\omega^2 r^2 w/g$$

$$P = A\omega^2 r^2 w/g \text{ or } Av^2 w/g \quad . \quad . \quad . \quad . \quad (4)$$

Or the intensity of stress, *i.e.* force per unit of cross-section,

$$P/A = w\omega^2 r^2/g \text{ or } wv^2/g \quad . \quad . \quad . \quad . \quad (5)$$

Evidently if g is in ft./sec.^{2}, w is in lb./ft.^{3} and v in ft./sec., P/A will be in lb./ft.^{2} But pounds per square *foot* is an unusual unit for a stress intensity. If P/A is to be in lb./sq. in., w must be in lb./cu. in., v in *inches*/sec. and g in *inches*/sec.^{2}, *i.e.* 32·2 × 12 in./sec.^{2} Then

$$P/A = wv^2/(32{\cdot}2 \times 12) \text{ or } w\omega^2 r^2/(32{\cdot}2 \times 12) \quad . \quad (6)$$

EXAMPLE. A wheel has a thin rim of steel 6 ft. diameter. At what speed in r.p.m. may it revolve without causing a stress greater than 7 tons per sq. in. if the steel weighs 0·28 lb. per cu. in.? Neglect any effect of spokes.

From (6)

$$P/A = 7 \times 2240 \text{ lb./in.}^2$$
$$= (0{\cdot}28 \text{ lb./in.}^3) \times \omega^2 \times 36^2 \text{ in.}^2/(32{\cdot}2 \times 12) \text{ in./sec.}^2$$
$$\omega^2 = (7 \times 2240 \times 32{\cdot}2 \times 12)/(0{\cdot}28 \times 36^2) \text{ (radian/sec)}^2$$
$$\omega = 129{\cdot}2 \text{ radians/sec. which is equal to 1234 r.p.m.}$$

EXAMPLES VI

1. At what speed will a locomotive produce a side thrust equal to $\frac{1}{50}$ of its own weight on the outer rail of a level curved railway line, the radius of the curve being 750 feet ?

2. What is the least radius of curve round which a truck may run on level lines at 20 miles per hour without producing a side thrust of more than $\frac{1}{100}$ of its own weight ?

3. How much must the outer rail of a line of 4 feet 8½ inches gauge be elevated on a curve of 800 feet radius in order that a train may exert a thrust normal to the track when travelling at 30 miles per hour ?

4. At what speed can a train run round a curve of 1000 feet radius without having any thrust on the wheel flanges when the outer rail is laid 1·5 inches above the inner one, and the gauge is 4 feet 8½ inches ?

5. To what angle should a circular cycle-track of 15 laps to the mile be banked for riding upon at a speed of 30 miles per hour, making no allowance for support from friction ?

6. What will be the inclination to the vertical of a string carrying a weight suspended from the roof of a railway carriage of a train going round a curve of 1000 feet radius at 40 miles per hour ?

7. The wheels on the same axle of a truck are 4 feet 6 inches apart and the c.g. of the truck and its load is 6 feet above rail level. If the truck travels round a level curve of 600 ft. radius at a speed of 30 m.p.h., find the normal (vertical) pressure on the inner and outer wheels if the loaded truck weighs 25 tons. What elevation of the outer rail would equalize the normal pressures on the inner and outer rails ? At what speed would the truck be upset with no elevation of the outer rail ? (Assume symmetry of truck and load).

8. A truck travels round a curve of 50 feet radius on an unbanked road. At what speed will it side-slip or skid if the coefficient of sliding friction between the tyre and the road is 0·3 ? If the wheel axles are 5 feet apart, what is the greatest height above road level of the c.g. of the loaded truck in order that it shall not overturn before the side-slipping speed is reached ? (Assume symmetry of truck and load.)

9. Solve No. 8 if the road is banked to an angle of 10°.

10. Find the height of a simple governor to run at 40 revolutions per minute neglecting friction. If the friction at the sleeve is equal in each direction to a force one-tenth of the weight of the two balls, find the upper and lower limits of the governor speed without alteration of the position of the sleeve.

11. If the governor in No. 10, *i.e.* of the same height, carries a load 5 times the weight of the two balls, find the speeds of running without friction and the limits without change of sleeve position if the friction at the sleeve can amount to $\frac{1}{10}$ of the weight of the two balls.

12. A loaded governor has arms and links all 12 inches long. The two balls each weigh 6 lb. and the central load is 30 lb. If, at the lower end of the speed range, the radius of the ball path is 7·5 in., and at the maximum speed it is 9 inches, find the extreme speeds if friction is neglected.

13. Three masses of 15, 12, and 10 lb. at radii of 10, 12, and 14 inches respectively revolve in the same plane. The second and third masses have their centres of gravity at angular positions of 90° and 240° respectively from that of the first. Find the magnitude and position of a mass at a radius of 20 inches to balance the three masses.

14. An unbalanced rotating mass is equivalent to 60 lb. at a radius of 2 feet from the axis of rotation. What two masses with their centres of gravity in the same axial plane, situated one in a plane of rotation distant 1 foot from that of the 60 lb. mass and the other 1·5 foot on the opposite side, will balance the system if both the added balance masses have their centres of gravity at a distance of 20 inches from the axis of rotation ?

15. Solve No. 14 if the two masses rotate in planes both on the same side of the plane of rotation of the 60 lb. which they are to balance.

16. Find the magnitude and position of the weights required on the wheels of a locomotive to balance the rotating parts if the centres of gravity of the balance weights are 5 feet apart and at 2·5 feet radius, the cylinders being symmetrically placed between the wheels with centre lines 2 feet apart. The cranks have a throw of 10 inches and the weight of the rotating parts are equivalent to 500 lb. at the axis of each crank pin.

17. Find the linear speed at which the rim of a cast iron fly-wheel may revolve if the resulting centrifugal tension is not to exceed 2000 lb. per sq. in., neglecting the effects of spokes. Take the weight of cast iron as 0·26 lb. per cu. in.

CHAPTER VII

OSCILLATORY MOTION

74. Simple Harmonic Motion. This is the simplest form of to and fro or reciprocating motion. The study of it can be approached in two ways; the first and most usual is to define the displacement of a point and then find its velocity and acceleration at any period in its repeated reciprocating motion. The second is to specify the varying force for the different displacements producing simple harmonic motion and find from it the kinetic energy, speeds, and accelerations at those displacements.

(1) *Simple Harmonic Motion as Projected Circular Motion.* If a point Q, Fig. 93, moves in a circular path AQB of

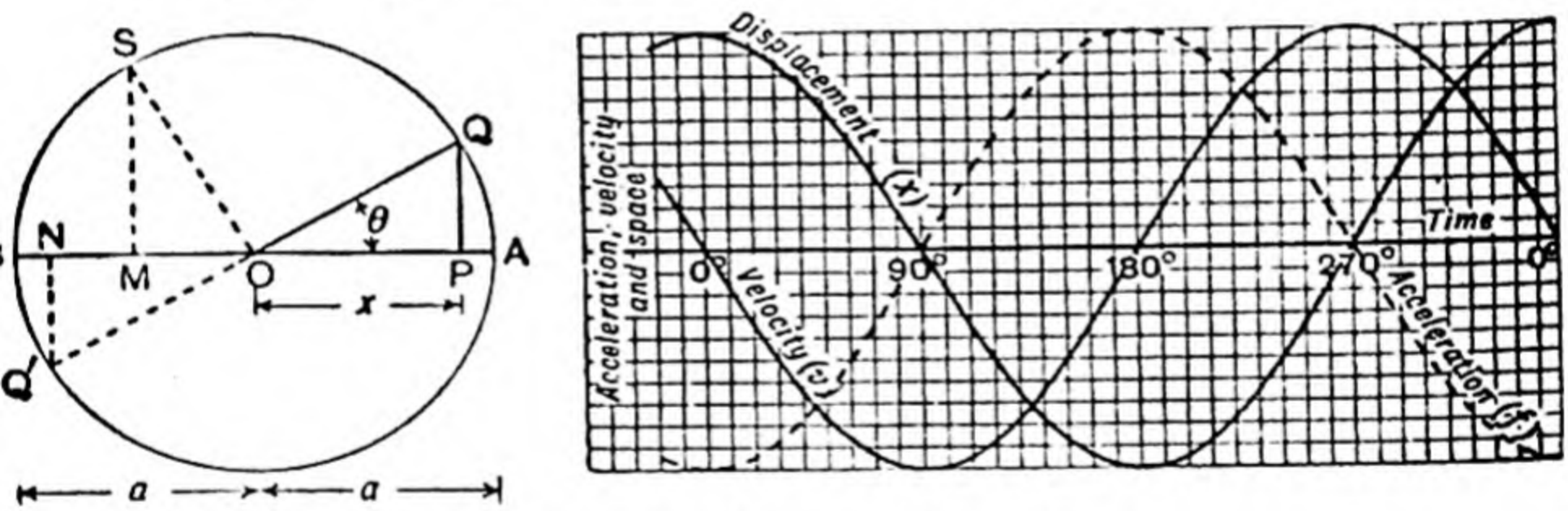

FIG. 93.—Simple harmonic motion.

radius a with constant angular velocity ω (or constant linear speed $a\omega$), then the motion of P, the rectangular projection of Q on a diameter AB, is simple harmonic motion. Let θ be any angular displacement $\widehat{AOQ}$ of Q and let x=OP be the corresponding linear displacement of P from its mid-stroke position at O. Then

$$x = a \cdot \cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The length a, the half stroke, is called the amplitude of the motion and θ gives its phase at P. A complete oscillation

or vibration is from A to B and back to A again, made while Q travels a complete circumference counter-clockwise from A through S, B, and Q′ to A again with constant angular velocity ω. If T is the time for a vibration, *i.e.* for the radius vector OQ to cover a circle or 2π radians,

$$T = 2\pi/\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and is in seconds if ω is in radians per second.

In any position such as Q, the speed of P is the horizontal projection of the velocity ωa of Q, *i.e.* $\omega a \times OM/OS$, for Q is moving parallel to OS, hence

$$\text{speed of P} = v = \omega a \times OM/OS = -\omega a \sin\theta \quad . \quad (3)$$

the negative sign being used to indicate that the direction of v is opposite to that in which x is positive or that $\omega a \sin\theta$ is the rate of *decrease* of x.

In terms of the displacement x, since $\sin^2\theta = 1 - \cos^2\theta$,

$$v = -\omega\sqrt{a^2 - x^2} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The acceleration of Q is $\omega^2 a$ along QO towards O, and the acceleration of P is the horizontal projection of this,

$$\textit{i.e.} \qquad f = -\omega^2 a \cos\theta = -\omega^2 x \quad . \quad . \quad . \quad . \quad (5)$$

the minus sign being used since f is directed towards O in the opposite direction to that in which x is positive. It indicates that v is decreasing with increase of x (*i.e.* the negative magnitude of v is increasing). Note that f is proportional to the displacement x and at the rate of ω^2 per unit of displacement. If x is in feet, ω^2 is the acceleration in ft./sec.^{2} per ft. of displacement or acceleration/displacement. Thus the time of a complete vibration or oscillation may be expressed as

$$T = 2\pi/\omega = 2\pi\sqrt{\text{displacement/acceleration}} \quad . \quad (6)$$

If we reckon time from the displacement a at A, $\theta = \omega t$ where t is time in seconds, and we may write,

$$\text{Displacement} = x = a\cos\omega t, \text{ velocity} = v = dx/dt = -a\omega\sin\omega t,$$
$$\text{Acceleration} = f = d^2x/dt^2 = -a\omega^2\cos\omega t.$$

The displacement of P is represented by the horizontal projection of the rotating vector OQ, the velocity of P by projection of the rotating vector OQ, the velocity of P by QP or OM (the latter being the horizontal projection of OS,

a rotating vector representing $-\omega a$), and the acceleration by ON the projection of a rotating vector OQ′ representing $-\omega^2 a$. Then we may write

Displacement, $x=a\cos\omega t$; velocity, $v=dx/dt$

$$=a\omega\cos(\omega t+\tfrac{1}{2}\pi),$$

$$\text{Acceleration, } f=d^2x/dt^2=a\omega^2\cos(\omega t+\pi).$$

these being the projections on the horizontal of the radii vectors OQ, OS and OQ′. Thus the displacement, velocity, and acceleration may be represented in related graphs on a base of time t or angle ωt by three cosine curves of different phase, as illustrated to the right in Fig. 93. The phase or angle of the velocity graph is $\frac{1}{2}\pi$ radians or 90° in front of that for the displacement graph, and that for the acceleration is π radians or 180° in front of the displacements. (Note that the three graphs represent the *horizontal* projections of Q, S, and Q′, and are not projected sideways from the circle in Fig. 93, which would give vertical projections, but from points 90° in advance of these points.)

The relations of displacement-time, velocity-time, and acceleration-time graphs to one another (Art. 48) are well illustrated in these curves, *e.g.* the maximum v occurs when the slope of the x graph is a maximum, *i.e.* when $x=0$, zero velocity when the slope of the x graph is zero, maximum acceleration when the slope of the v graph is greatest, zero acceleration when the slope of the v graph is zero, and so on.

Forces producing Simple Harmonic Motion. From the known value of the acceleration (5) we know the variable force P necessary to give simple harmonic motion to a body of weight W, say, viz.

$$\mathrm{P}=\mathrm{W}f/g=-\mathrm{W}\omega^2x/g \text{ or } -ex \quad . \quad . \quad (7)$$

where $e=$ force per unit displacement$=\mathrm{W}\omega^2/g$

Acceleration $\omega^2x=-\mathrm{P}g/\mathrm{W}$

or

$$\omega^2=-\mathrm{P}g/\mathrm{W}x=-(\mathrm{P}/x)\div(\mathrm{W}/g) \quad . \quad . \quad (8)$$

the negative sign merely indicating that P is of opposite sign to x. We may write (8) as

$$\omega=\sqrt{(\text{force per ft. of displacement/mass})} \quad . \quad (9)$$

A force P, proportional to the displacement from mid-stroke, as in (7), is provided by the force of a strained elastic spring, and oscillations of a body controlled by a spring are simple harmonic. If e is the force exerted by a spring per foot of displacement, $P=ex$, where x is in feet, and (9) becomes

$$\omega=\sqrt{eg/W} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and the time of a complete vibration from (2) is

$$T=2\pi\sqrt{W/eg} \text{ or } 2\pi\sqrt{\delta/g} \quad . \quad . \quad . \quad (11)$$

where $\delta=W/e$ is the displacement due to a static load W.

(2) *Energy in Simple Harmonic Motion.* If instead of specifying the motion we specify the forces which produce simple harmonic motion, we can find the acceleration and displacement as follows. Suppose a body of weight W has exerted upon it, by a spring, a force proportional to the displacement from an equilibrium position of the spring,

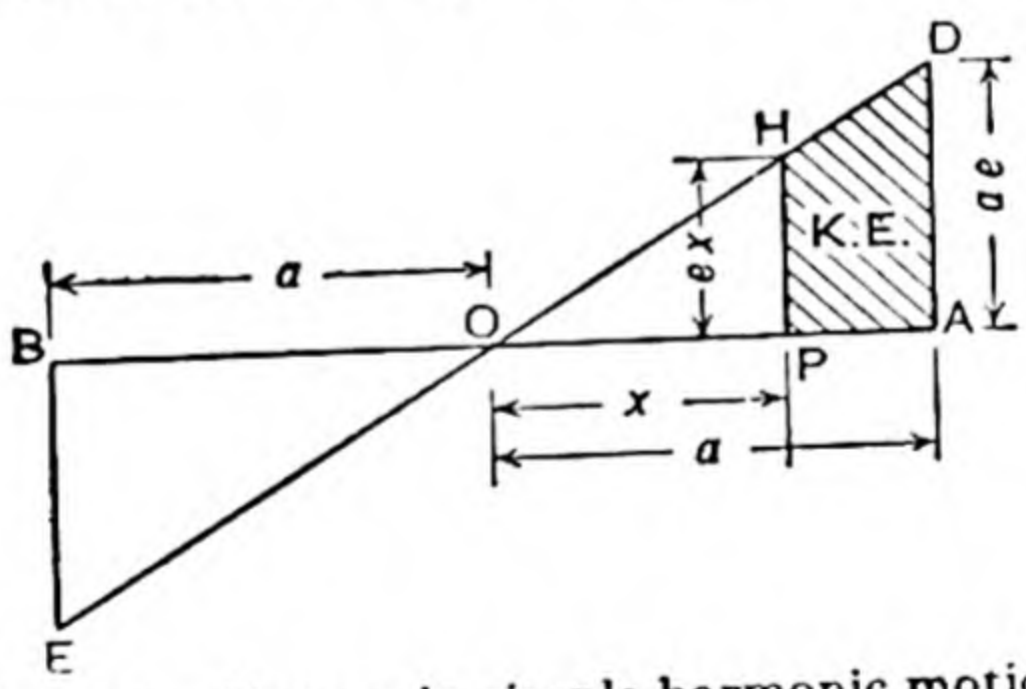

FIG. 94.—Energy in simple harmonic motion.

at the rate of e lb. (force) per ft. of displacement and that it has a vibration amplitude a. Then we can draw the diagram of work done in stretching the spring. At $x=0$, the force $=0$. At displacement x, it is ex rising to a maximum ae at the end of the stroke. The diagram of work done in stretching the spring is as shown in Fig. 94. The work done in straining the spring to the full amplitude a is represented by the triangle OAD or $\frac{1}{2}a\times ae=\frac{1}{2}ea^2$. At

mid-stroke O, this has become wholly kinetic energy so that

$$\text{total energy} = \tfrac{1}{2}ea^2 \quad . \quad . \quad . \quad . \quad (12)$$

When the c.g. of the body is at P, *i.e.* at a distance x from O, the energy stored in the spring is $\frac{1}{2}ex^2$ represented by the area of the triangle OPH, and the remaining energy is kinetic, which is represented by the trapezium PADH: its amount is

$$\text{K.E.} = \tfrac{1}{2}ea^2 - \tfrac{1}{2}ex^2 = \tfrac{1}{2}e(a^2 - x^2) = \tfrac{1}{2}\mathrm{W}v^2/g \quad . \quad (13)$$

Hence

$$v^2 = ge(a^2 - x^2)/\mathrm{W} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$v = \sqrt{ge(a^2 - x^2)/\mathrm{W}} \quad . \quad . \quad . \quad . \quad (15)$$

And at $x = 0$

$$v_0 = \sqrt{gea^2/\mathrm{W}} = a\sqrt{ge/\mathrm{W}} \quad . \quad . \quad (16)$$

where v_0 is the speed at O for $x = 0$,

and

$$v/v_0 = \sqrt{(a^2 - x^2)}/a \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The values of v from this or (15), if plotted on x as a base line, give a circle of radius a representing v_0 as graph, and the ratio (17) is that between one side $\sqrt{a^2 - x^2}$ and the hypotenuse a of a right-angled triangle, the third side of which is x. Or the ratio is sin θ where $x/a = \cos\theta$. Thus

$$v/v_0 = \sin\theta \quad \text{or} \quad v = v_0 \sin\theta \quad . \quad . \quad . \quad (18)$$

which is the projection of a speed v_0 along the circumference on a diameter AB of a circle of radius a, Fig. 95. Since $v = 0$ at $x = a$, the times of travel of a point along a diameter AB, Fig. 95, will correspond with those of a point vertically above, moving at a constant speed v_0 along the circumference. In other words, the c.g. of the body will coincide with the projection P of the point moving in the circumference of the circle with constant speed v_0. Thus we reach the original specification of the motion from consideration of the force producing it and the distribution of the energy between K.E. and strain energy in the spring which supplies and regulates the driving effort.

Calculus Notation. The relations of x, v, and f are very

simple. If we write $\theta=\omega t$, $d\theta/dt=\omega$(constant); also $dx/dt=v$, and d^2x/dt^2 or $dv/dt=f$, then

$$x=a\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$v=dx/dt=(dx/d\theta)\times d\theta/dt=-a\omega\sin\theta \quad . \quad (20)$$

$$f=dv/dt=\frac{dv}{d\theta}\cdot\frac{d\theta}{dt}=-a\omega^2\cos\theta=-\omega^2x \quad . \quad (21)$$

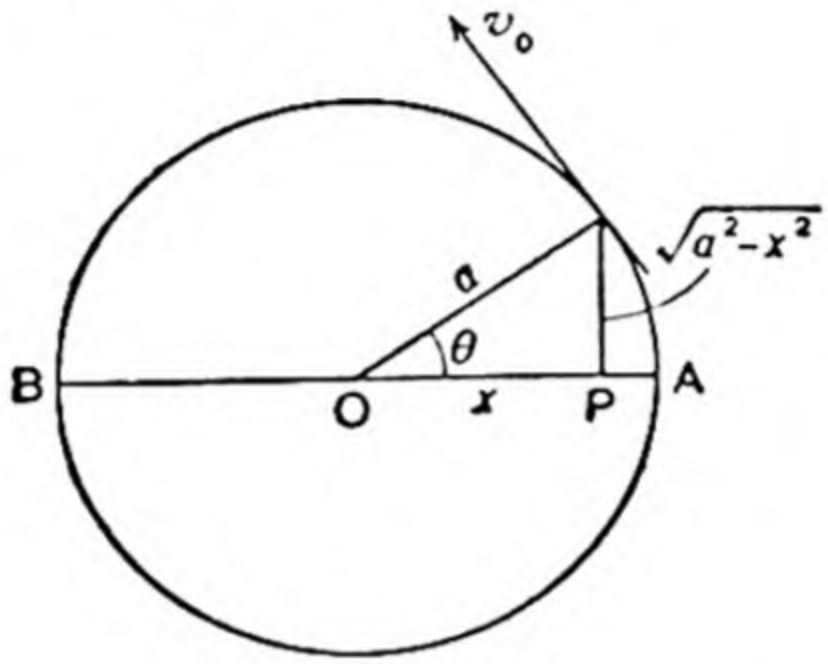

FIG. 95.

EXAMPLE 1. A weight rests freely on a scale-pan of a spring balance, which is given a vertical simple harmonic vibration of period 0·5 second. What is the greatest amplitude the vibration may have in order that the weight may not leave the pan? What is then the pressure of the weight on the pan in its lowest position?

Let a=greatest amplitude in feet.

The greatest downward acceleration on the body is g, occurring when the weight is in its highest position and the spring is about to return. Hence, if the scale-pan and weight do not separate, the downward acceleration of the pan must not exceed g. But the greatest acceleration is ω^2a where $\omega=2\pi/T=2\pi/0{\cdot}5$, therefore ω^2a must not exceed g, or

$a\times(2\pi/0{\cdot}5)^2$ must not exceed g

a must not exceed $g\times(0.5/2\pi)^2$

or 32·2 (ft./sec.^{2}) $\times(0{\cdot}5/2\pi)^2$ sec.^{2}

a „ „ 0·204 ft. or 2·45 inches.

If the balance has this amplitude of vibration, the pressure between the pan and weight at the lowest position will be equal to twice the weight, since there is an acceleration g upwards which must be caused by an effective force equal to the weight, acting upwards, or a gross pressure of twice the weight, from which the downward gravitational force has to be subtracted.

EXAMPLE 2. Part of a machine has a reciprocating motion, which is simple harmonic in character, making 200 complete oscillations in a minute; it weighs 10 lb. Find (1) the accelerating force upon it in pounds and its velocity in feet per second, when it is 3 inches from mid-stroke; (2) the maximum accelerating force; and (3) the maximum velocity if its total stroke is 9 inches, *i.e.* if its amplitude of vibration is 4·5 inches.

$$\text{Time of 1 oscillation} = 60/200 = 0{\cdot}3 \text{ second}$$

$$\left.\begin{array}{l}\text{therefore the acceleration per foot}\\ \text{distance from mid-stroke}\end{array}\right\} = (2\pi/0{\cdot}3)^2 \text{ ft./sec.}^2$$

and the accelerating force 0·25 ft. from mid-stroke on 10 lb. is

$$(10/32{\cdot}2) \times 0{\cdot}25 \times (2\pi/0{\cdot}3)^2 = 34.06 \text{ lb.}$$

and the maximum accelerating force 4·5 inches from mid-stroke is 1·5 times as much as at 3 inches, namely, 51·09 lb.

The maximum velocity in ft. per sec. occurring at mid-stroke $= (\omega a) =$ amplitude in ft. $\times 2\pi/0{\cdot}3 = (4{\cdot}5/12) \times 2\pi/0{\cdot}3 = 7{\cdot}85$ ft./sec.

Velocity at 3 inches from mid-stroke

$$= 7{\cdot}85 \times \sqrt{4{\cdot}5^2 - 3^2}/4{\cdot}5 = 5{\cdot}85 \text{ ft./sec.}$$

EXAMPLE 3. A light helical spring is found to deflect 0·4 inch when an axial load of 4 lb. is hung on it. How many vibrations per minute will this spring make when carrying a weight of 15 lb. ?

The force per foot of deflection $= 4 \div 0{\cdot}4/12 = 120$ lb.

Time of vibration $T = 2\pi\sqrt{15/(32{\cdot}2 \times 120)} = 0{\cdot}3914$ second.

And the number of vibrations per minute is 60/0·3914 =153·3.

75. Reciprocating Piston Motion. The reciprocating motion in a straight line of a piston or a crosshead attached to a connecting rod, the other end of which moves at constant speed in a circle, about the axis of the crankshaft, approximate to simple harmonic motion if the rod is long in comparison with the throw of the crank. The obliquity of a short rod modifies the motion to an extent depending on how short the rod is. Fig. 96 shows the crank and con-

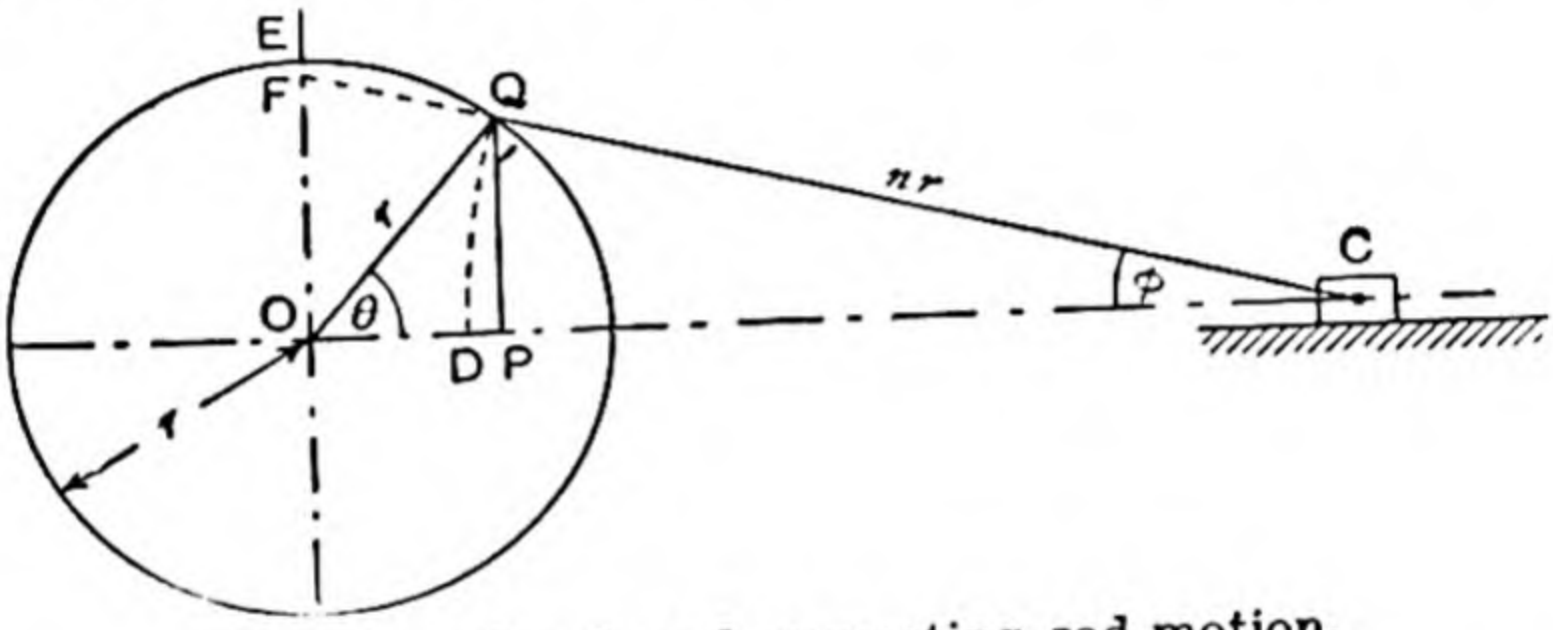

FIG. 96.—Crank and connecting rod motion.

necting rod mechanism for a very short rod QC, n times the length r of the crank OQ. When the crank is at an angle θ to the line OC, the displacement x, say, of the piston (or of the crosshead at C) from mid-stroke at O would be OP or $r \cos \theta$ for an indefinitely long rod, but is OD for this short rod. (QD is an arc of a circle with centre C.)

From Fig. 96 it is easy to find the velocity v of C as in Art. 52 by a vector velocity diagram, or by using the instantaneous centre of the rod QC. In either case, if ω is the angular velocity of Q and ωr its linear velocity, $v/\omega r$ =OF/OQ, or, $v=\omega r \times$ OF/OQ, since OQF is a vector velocity triangle for Q and C, with sides perpendicular to the velocities they represent; and we could find the acceleration of C from a speed-time graph of v and t or by a special geometrical construction. An easy approximate method of finding the velocity and acceleration of the piston (or crosshead C) is as follows.

It is easy to show [1] that if n is fairly large, the displacement from mid-stroke is approximately

$$x = \text{OD} = r\{\cos\theta - (1/2n)\sin^2\theta\} \quad . \quad . \quad (1)$$

Hence by differentiation with respect to time t, since $d\theta/dt = \omega$, a constant angular velocity of Q,

$$v = dx/dt = (dx/d\theta) \times (d\theta/dt)$$
$$= -\omega r\{\sin\theta + (1/2n)\sin 2\theta\} \quad . \quad (2)$$

$$f = -dv/dt = (dv/d\theta)(d\theta/dt)$$
$$= -\omega^2 r\{\cos\theta + (1/n)\cos 2\theta\} \quad . \quad (3)$$

At $\theta = 0$, this becomes $-\omega^2 r(1 + 1/n)$; and at $\theta = 180°$, its value is $+\omega^2 r(1 - 1/n)$. It reaches zero before $\theta = 90°$ (the angle for zero acceleration when n is infinte, as in simple harmonic motion), viz. when $\cos\theta = -(1/n)\cos 2\theta$, and is $+\omega^2 r/n$ when $\theta = 90°$, and a smooth curve through these four points (or even three for $\theta = 0°$, 90°, and 180°) would give an approximate graph.

EXAMPLE 1. The crank of an engine makes 150 revolutions per minute, and is 1·3 feet long. It is driven by a piston and a very long connecting rod, so that the motion of the piston may be taken as simple harmonic. Find the piston speed and the force necessary to accelerate the piston and other reciprocating parts weighing altogether 300 lb. (1) When the piston is farthest from the crankshaft. (2) When the crank has turned through 45° from the first position. (3) When the piston has moved 0·65 ft. from the first position. (4) and (5) When the crank has turned through 90° and 180° respectively from its first position.

The angular velocity ω of the crank is

$$150 \times 2\pi/60 = 5\pi \text{ or } 15{\cdot}71 \text{ radians per sec.}$$

Linear velocity of crankpin $= 5\pi \times 1{\cdot}3 = 19{\cdot}59$ ft./sec.

Using Fig. 93 and equations (1), (3), and (7), Art. 74,

[1] $\text{DP} = nr(1 - \cos\phi) = nr(1 - \sqrt{1 - \sin^2\phi})$ and since $\text{QP} = r\sin\theta = nr\sin\phi$, $\text{DP} = nr(1 - \sqrt{1 - (1/n^2)\sin^2\theta} = r \times (1/2n)\sin^2\theta$ approximately.

(1) When $\theta=0$,

$x=1{\cdot}3\cos 0°=1{\cdot}3$ ft.

$v=-19{\cdot}59\sin 0°=0$

$f=-\omega^2 r\cos 0°=-25\pi^2\times 1{\cdot}3\times 1=-320{\cdot}7$ ft./sec.^{2},

accelerating force

P $=300$ lb. $\times 320{\cdot}7/32{\cdot}2=2987$ lb. (toward centre).

(2) When $\theta=45°$,

$x=1{\cdot}3\times\cos 45°=1{\cdot}3\times 0{\cdot}7071=0{\cdot}919$ ft.

$v=-19{\cdot}59$ ft./sec. $\times 0{\cdot}7071=-13{\cdot}86$ ft./sec.

$\mathrm{P}=-320{\cdot}7\times 0{\cdot}7071\times 300/32{\cdot}2$

$=2113$ lb. toward centre.

(3) When $x=1{\cdot}3-0{\cdot}65=0{\cdot}65$ ft., $\cos\theta=0{\cdot}65/1{\cdot}3=0{\cdot}5$
$\theta=60°$,

$v=19{\cdot}59\sin 60°=19{\cdot}59\times 0{\cdot}866=16{\cdot}96$ ft./sec.

$\mathrm{P}=-300\times 320{\cdot}7\times 0{\cdot}5/32{\cdot}2=-2987/2$

$=1493$ lb. toward centre.

(4) When $\theta=90°$,

$x=0$, $\mathrm{P}=0$

$v=19{\cdot}59\sin 90°=15{\cdot}59$ ft./sec.

(5) When $\theta=180°$,

$x=1{\cdot}3\times(-1)=-1{\cdot}3$ ft.

$v=0$

P $=-2987$ lb. $\times(-1)=2987$ lb. (towards centre).

EXAMPLE 2. Work Example 1 when the very long rod is replaced by one four times as long as the crank ($n=4$).

We can find the piston velocities by vector velocity diagrams, but the approximate expression (2), Art. 75, will suffice and the accelerating forces in lb. (force) will be 300/32·2 times the acceleration in ft./sec.^{2} given by (3), Art. 75. We can use many of the arithmetical results of Example 1.

Using Fig. 96, quantities directed to the right will be positive and those to the left negative.

(1) When $\theta = 0°$ (Fig. 96)

$$x = 1{\cdot}3 \text{ ft.}$$

and from (2)

$$v = -19{\cdot}59 \times 0 = 0$$

From (3)

$$P = -320{\cdot}7(1 + 0{\cdot}25) \times 300/32{\cdot}2 = -2987 \times 1{\cdot}25$$
$$= 3734 \text{ lb. (to centre).}$$

(2) When $\theta = 45°$

$$x = 1{\cdot}3(0{\cdot}7071 - 0{\cdot}125 \times 0{\cdot}5) = 0{\cdot}8375 \text{ ft.}$$
$$v = -19{\cdot}59(0{\cdot}7071 + 0{\cdot}125)$$
$$= -16{\cdot}3 \text{ ft./sec. (to centre).}$$
$$P = -320{\cdot}7(0{\cdot}7071 + 0) \times 300/32{\cdot}2$$
$$= 2113 \text{ lb. to centre.}$$

(3) $x = 0{\cdot}65$ ft. $= 1{\cdot}3 - 1{\cdot}3(\cos\theta - 0{\cdot}125 \sin^2\theta)$ and dividing by 1·3,

$$1 - \cos\theta + 0{\cdot}125 \sin^2\theta = 0{\cdot}5$$
$$-\cos\theta + 0{\cdot}125(1 - \cos^2\theta) = -0{\cdot}5$$

multiplying by 8,

$$\cos^2\theta + 8\cos\theta - 5 = 0$$

Solving this quadratic equation, $\cos\theta = -4 + \sqrt{21} = 0{\cdot}583$, $\theta = 54{\cdot}3°$

$$v = -19{\cdot}59\ (0{\cdot}8121 + 0{\cdot}125 \sin 108{\cdot}6°)$$
$$= -19{\cdot}59 \times 0{\cdot}9305 = -18{\cdot}23 \text{ ft./sec.}$$
$$P = -300 \times 320{\cdot}7(\cos 54{\cdot}3° + 0{\cdot}25 \cos 108{\cdot}6°)/32{\cdot}2$$
$$= -2987(0{\cdot}5835 - 0{\cdot}0797) = 1504 \text{ lb. (to centre).}$$

(4) When $\theta = 90°$,

$$x = 1{\cdot}3(0 - 0{\cdot}125) = -0{\cdot}1625 \text{ ft.}$$
$$v = -19{\cdot}59(1 + 0) = 19{\cdot}59 \text{ ft./sec. as in Ex. 1.}$$
(Note that in Fig. 96 F would fall on E.)
$$P = -2987(0 - 0{\cdot}25) = +747 \text{ lb.}$$

(5) When $\theta = 180°$,

$x = -1{\cdot}3$ ft.

$v = 0$

$P = -2987(-1 + 0{\cdot}25) = +2240$ lb. (toward centre).

The graphs of v and f on a base of angle (or time) are shown in Fig. 97 with the corresponding graphs for a very long rod (*i.e.* for simple harmonic motion) in broken lines for comparison. The effect of the obliquity of the connect-

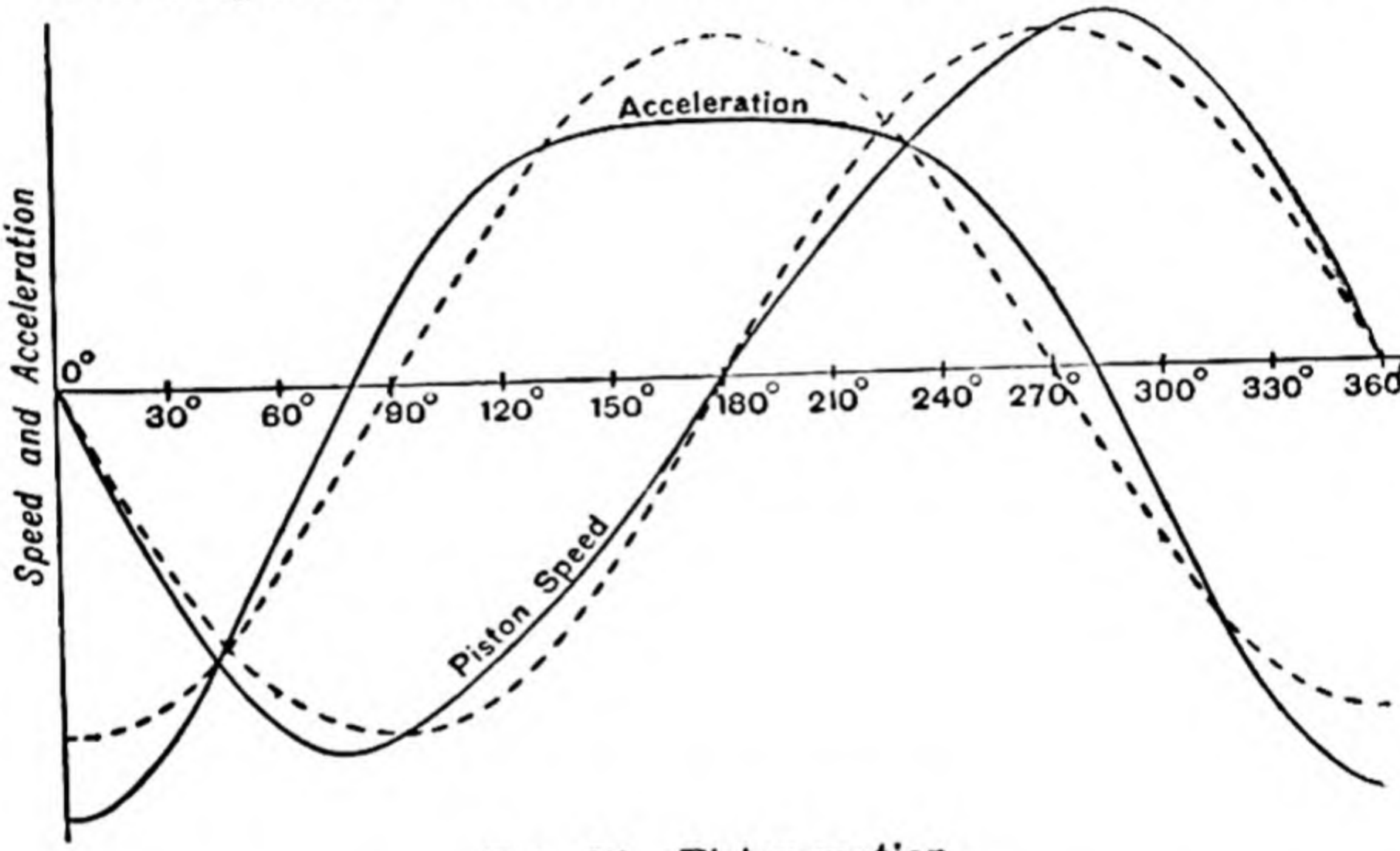

FIG. 97.—Piston motion.

ing rod in modifying the sine and cosine curves of simple harmonic motion is considerable. Note that the values of v are equal at $\theta = 90°$, and from 90° to 180° the values are smaller for connecting-rod mechanism than for simple harmonic motion; it will be obvious from Fig. 96 that F in this range will fall lower than Q, though above it from $\theta = 0°$ to $\theta = 90°$.

In Fig. 98 the acceleration f is shown on a base of values of x the piston displacement. If x is measured from the right-hand end of the stroke, as indicated on the reduced diagram of the mechanism, then the accelerations at that end are positive as v is increasing with time, and both

acceleration and displacement (x) are positive to the left and ordinates of acceleration are graphed as positive upwards. The area ZAD between the graph and the base line represents kinetic energy stored in the reciprocating parts during the acceleration from the beginning of the stroke at A to the point Z, where the acceleration is zero. And the area ZEB represents the energy restored from the

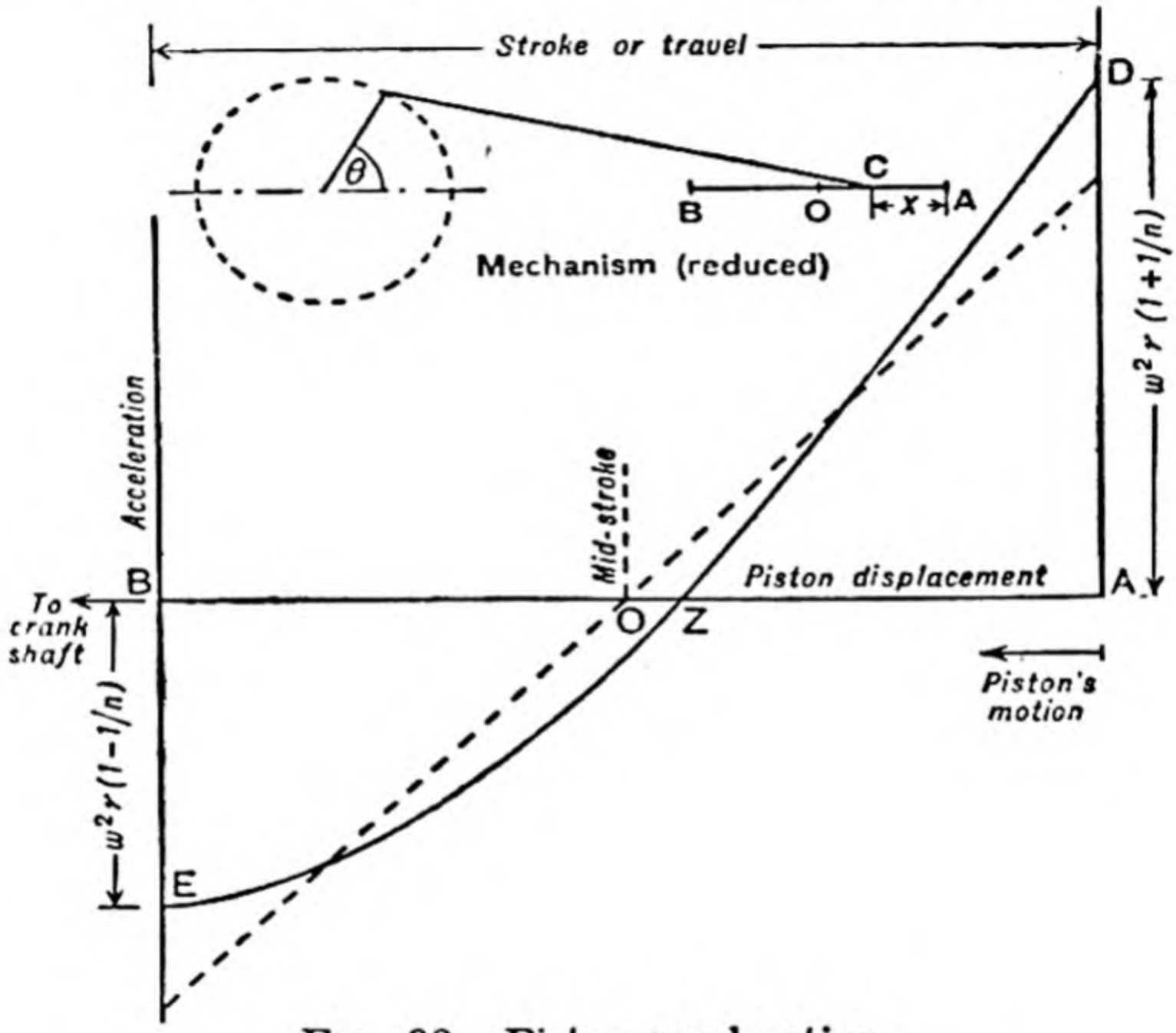

FIG. 98.—Piston acceleration.

reciprocating parts to the crankshaft during the period in which the piston is decelerating and losing K.E. The areas ZAD and ZEB are, of course, equal.

The straight line graph shown by a broken line indicates the corresponding accelerations for simple harmonic motion.

The graph in Fig. 98 is of some importance; for nearly all purposes it can be drawn with sufficient accuracy by finding a few points, including those for the two end ordinates. To a different scale, the same curve represents

P, the force required to accelerate the reciprocating parts which is W/g times the acceleration f. These forces (P) represent diminution of the piston thrust transmitted to the crank in the first part of the stroke and addition to it in the latter part of the stroke, when the sign of the acceleration has changed. Thus a knowledge of the force required to accelerate the piston, crosshead, and other reciprocating parts is necessary to determine the turning moment exerted on the crankshaft in the different positions of the mechanism during a revolution of the crankshaft. But the movement of the connecting rod and its linear and angular acceleration is less simple. A portion is often taken as reciprocating and another part as rotating at the crank pin, an assumption which is often justified as near enough but is not exact. So it is reasonable also to use the simple approximation given in this article for the acceleration of the reciprocating parts.

Crank Effort Diagram. In order to estimate the fluctuation of energy within one revolution of an engine crankshaft, a crank effort diagram is plotted. In this, the horizontal ordinates may be distances along the circumference of the circle described by the centre of the crank pin, and vertical ordinates the resolved forces perpendicular to the crank which result from the thrust of the piston acting through the connecting rod. Neglecting friction, knowing the thrust exerted by the piston, we can find the resolved force perpendicular to the crank at the crank pin by, say, the principle of work. In Fig. 96, if P_p is the horizontal force exerted on the connecting rod by the piston (or the crosshead) and R is the resolved component force perpendicular to the crank at Q, ωr the crank pin speed and v_p the piston speed, then from (2), Art. 46,

$$v_p \times P_p = R \times \omega r \quad \text{or} \quad R = P_p \times v_p/\omega r \quad . \quad (4)$$

But from Art. 75,

$$v_p/\omega r = OF/OQ, \text{ therefore } R = P_p \times OF/OQ \quad . \quad (5)$$

The quantity P_p will be total gas or vapour pressure on the piston minus the force required to accelerate the reciprocating parts. In plotting such a crank effort diagram, we should first plot the total effective pressure on the piston with the stroke as a base and then plot the

accelerating force curve, similar to Fig. 98, on the same scale and use the difference of the ordinates as the effective pressure on the gudgeon pin at the small end of the connecting rod. This is the force P_p above. The crank effort diagram, in conjunction with a curve of resisting force (reduced to its equivalent at the crank pin, *i.e.* the resisting torque divided by the length of the crank throw), enables us to determine the excess energy to be stored and restored by the flywheel, the quantity δE of Art. 67. This energy δE is represented by the loops made by the successive intersection of the curves of (1) crank effort and (2) resistance. If the resistance is constant it is equal to the total energy of one revolution divided by the circumference of the crank-pin circle. To a different scale, the curves of crank effort and resisting force represent driving torque and resisting torque. The construction of a crank effort diagram and determination of δE make an instructive exercise for the drawing office. In a fast-running reciprocating engine the form is materially affected by the inertia of the reciprocating parts, which take considerable forces to accelerate and decelerate them.

Piston Speed-displacement Graph. The piston speed-time curve on Fig. 97 may be replotted on a distance or piston displacement base. This graph for simple harmonic motion is a semicircle ($v \propto \sin\theta$ and $x \propto \cos\theta$), while that for the piston or crosshead with crank and connecting rod motion is a somewhat similar curve but lop-sided, going outside the semicircle in the first part of the stroke and falling inside it in the latter part (as is evident from Fig. 97). The graph is not of much importance.

EXAMPLE 3. If the engine of Example 2 has a cylinder 6 in. diameter, find the crank effort when the crank angle, θ, is 45° if the effective steam pressure on the piston is 150 lb. per sq. in.

Total effective pressure on piston is 150 lb./sq. in. $\times 6^2 \times 0{\cdot}7854$ sq. in.	$=4241$ lb.
From Ex. 2, at $\theta = 45°$, force to accelerate reciprocating parts	$=2113$ lb.
Force transmitted to connecting rod, by difference	$=2128$ lb.

$v_p = 16{\cdot}3$ ft./sec., $\omega r = 19{\cdot}59$ ft./sec., ratio $v_p/\omega r = 0{\cdot}8321$.

Hence from (4), crank effort (R)$=2128\times0{\cdot}8321=1770$ lb.

i.e. a driving torque of 1·3 ft. $\times1770$ lb. $=2301$ lb.-ft.

76. Torsional Simple Harmonic Motion. Simple harmonic motion is not necessarily rectilinear. A common form of it is a torsional oscillation, *i.e.* angular oscillation about an axis. The analogy between this and the linear simple harmonic motion of Art. 74 is very close. We may well start from the point of the forces operating during the oscillation.

Simple harmonic angular oscillation about an axis occurs when the moment of the forces exerted on a body about an axis is proportional to the angular displacement of the body from its mid-position of oscillation. If the angular position is specified as an angular displacement ϕ radians from the central position (or position of rest), then the moment of the forces exerted will be $M\phi$ where M is the moment for unit angular displacement (one radian) of the body or the moment of the forces about the axis of oscillation *per radian* of angular displacement from mid-position. (M is analogous to e in Art. 74 and ϕ to x.) Let ϕ_1 be the extreme value of ϕ or the angular amplitude of oscillation (analogous to a in rectilinear S.H.M., Art. 74).

If α is the angular acceleration of the body when it has the displacement ϕ,

$$(Wk^2/g)\alpha=-M\phi=(Wk^2/g)\omega^2\phi \quad . \quad . \quad . \quad (1)$$

corresponding to $(W/g)f=-ex$ in (7), Art. 74. The negative sign indicates that the angular speed is diminishing with increase of ϕ,

$$\alpha=\omega^2\phi=-M\phi/(Wk^2/g) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\omega^2=-M/(Wk^2/g) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\omega=\sqrt{\frac{\text{moment per radian displacement}}{(W/g)k^2 \text{ about axis}}} \quad . \quad (4)$$

$$T=\text{time of oscillation}=2\pi/\omega=2\pi\sqrt{Wk^2/Mg} \quad . \quad (5)$$

(note that here M does *not* denote mass). In this, ω is not to be confused with the variable angular speed $(d\phi/dt)$ of the body It is the constant angular velocity of a point

moving in a circle which specifies the motion. The circle would have as radius ϕ_1, *i.e.* a length equal to an arc subtending an angle ϕ_1 radians at the centre of a circle of unit radius. As in the linear simple harmonic motion, we should have,

$$\phi = \phi_1 \cos \theta \quad . \quad . \quad . \quad . \quad . \quad (6)$$

corresponding to (1), Art. 74, for rectilinear S.H.M. and the angular speed, say, ω' at this phase of the motion would be

$$\omega' = -\omega\phi_1 \sin \theta \quad . \quad . \quad . \quad . \quad . \quad (7)$$

corresponding to (3), Art. 74, for linear S.H.M., while angular acceleration would be

$$\alpha = -\omega^2\phi_1 \cos \theta = -\omega^2\phi \quad . \quad . \quad . \quad . \quad (8)$$

corresponding to (5), Art. 74, for linear S.H.M.

There is the same type of distribution between kinetic energy and potential energy (viz. elastic strain energy) as in linear S.H.M., and Fig. 94 would hold good if the base line represents values of ϕ, with total length $2\phi_1$ (in place of $2a$), and the vertical ordinates represent values of $M\phi$ (in place of ex). The area representing the K.E., viz. $\frac{1}{2}(Wk^2/g)(\omega')^2$, is the trapezium HDAP. The triangle OHP represents the (potential) elastic strain energy $\frac{1}{2}M\phi^2$ for an angular displacement OP or ϕ, the total energy being represented by the triangle OAD or $\frac{1}{2}M\phi_1{}^2$ or $\frac{1}{2}(Wk^2/g)(\omega\phi_1)^2$.

EXAMPLE. A metal disc is 10 inches diameter and weighs 6 lb. It is suspended from its centre by a vertical wire so that its plane is horizontal, and then twisted. When released, how many oscillations will it make per minute if the rigidity of the suspension wire is such that a twisting moment of 1 lb.-foot causes an angular deflection of 10° ?

The twisting moment per radian twist is $1/(10 \times \pi/180)$ $= 5{\cdot}73$ lb.-ft.

$k^2 = \frac{1}{2}(5/12)^2 = 0{\cdot}0868$ ft.^{2}

Time of vibration is $2\pi\sqrt{6 \times 0{\cdot}0868/(5{\cdot}73 \times 32{\cdot}2)} = 0{\cdot}334$ sec.

Vibrations per minute $= 60/0{\cdot}334 = 180$.

77. Pendulums. *Simple Pendulum.* This name refers strictly to a particle of indefinitely small dimensions and yet having weight, suspended by a perfectly flexible weightless thread from a fixed point, about which, as a centre, it swings freely in a circular arc. In practice, a small piece of heavy metal, usually called a pendulum bob, suspended by a moderately long thin fibre, behaves very nearly indeed like the ideal pendulum defined above, the resistances, such as that of the atmosphere, being small.

Let O, Fig. 99, be the point of suspension of the particle P of a simple pendulum.

Let OP, the length of thread, be l feet.

Let θ=angle $\widehat{AOP}$ in radians which OP makes with the vertical (OA) through O in any position P of the particle.

Draw PT perpendicular to OP, *i.e.* tangent to the arc of motion to meet the vertical through O in T.

The tension of the thread has no component along the direction of motion (PT) at P. The acceleration along PT is then $g \sin \theta$, since PT is inclined θ to the horizontal. If θ is very small, $\sin \theta$ may be taken equal to θ in radians. (If θ does not exceed 5°, the greatest error in this approximation is less than 1 part in 800.)

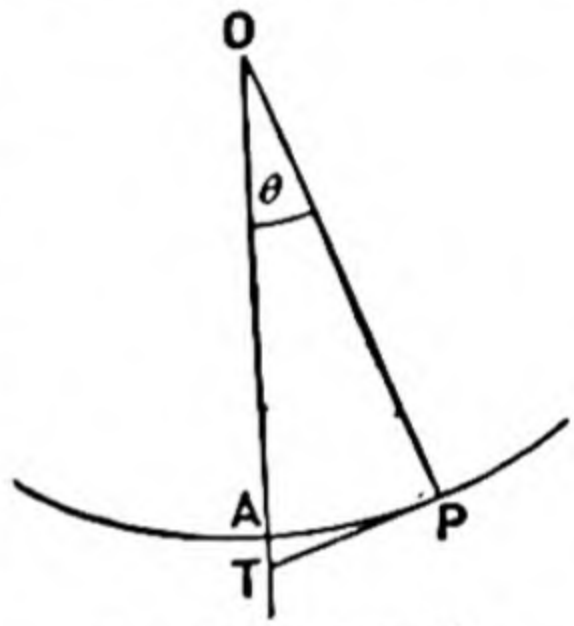

FIG. 99.—Simple pendulum.

Hence the acceleration along PT is $g\theta$ approximately. And θ=arc AP/radius OP; therefore acceleration along PT=$g \times$arc AP/l, and the acceleration is proportional to the distance AP, along the arc, of P from A, being g/l per foot of arc. Hence from (6), Art. 74, the time of a complete oscillation in seconds is

$$2\pi/\sqrt{g/l}=2\pi\sqrt{l/g} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the velocity at any point may be found, as in Art. 74, for any position of the swinging particle.

In an actual pendulum, the pendulum bob has finite dimensions, and the length l will generally be somewhat greater than that of the fibre by which it is suspended. The ideal simple pendulum having the same period of swing

as an actual pendulum of any form is called its *simple equivalent pendulum.* For this ideal pendulum the relation $t=2\pi\sqrt{l/g}$ holds, and therefore $l=\frac{1}{4}t^2g/\pi^2$, from which its length in feet may be calculated for a given time, t, of vibration.

The value of the acceleration of gravity, g, varies at different parts of the earth's surface, and the pendulum offers a direct means of measuring the value of this quantity g, viz. by accurate timing of the period of swing of a pendulum of known length. The length of an actual pendulum, *i.e.* of its simple equivalent pendulum, can be calculated from its dimensions.

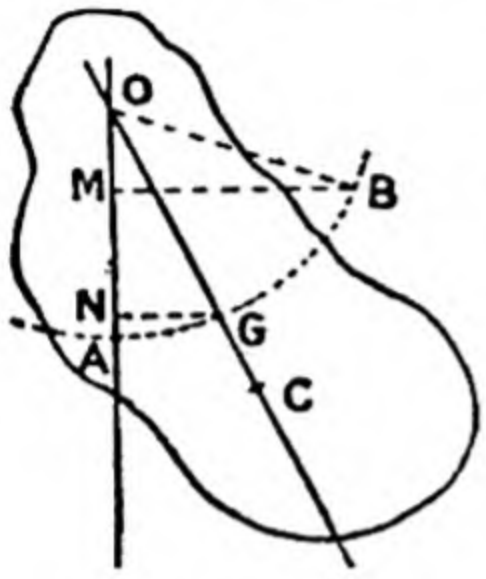

FIG. 100.—Compound pendulum.

Compound Pendulum. We now proceed to find the simple pendulum equivalent (in period) to an actual pendulum.

Let a body be suspended by means of a horizontal axis O (Fig. 100) perpendicular to the figure and passing through the body. Let G be the c.g. of the body in any position, and let OG make any angle θ with the vertical plane (OA) through O.

Suppose that the body has been raised to such a position that G was at B, and then released. Let the angle $\widehat{AOB}$ be ϕ, and $OG=OB=OA=h$.

The body oscillating about the horizontal axis O constitutes a pendulum.

Let l = length of the simple equivalent pendulum;

k_0 = radius of gyration about O;

k_g = radius of gyration about a parallel axis through G.

Let W be the weight of the pendulum, and let M and N be the points in which horizontal lines through B and G respectively cut OA.

When G has fallen from B to G, the work done is

$$W\times MN=W(ON-OM)=W(h\cos\theta-h\cos\phi)$$
$$=Wh(\cos\theta-\cos\phi)$$

Let the angular velocity of the pendulum in this position

be ω, then its kinetic energy is $\frac{1}{2}Wk_0^2\omega^2/g$ (Art. 64), and by the principle of conservation of energy (Arts. 46 and 63), if there are no resistances to motion the kinetic energy is equal to the work done, or

$$\tfrac{1}{2}Wk_0^2\omega^2/g = Wh(\cos\theta - \cos\phi) \quad . \quad . \quad (2)$$

and therefore

$$\omega^2 = 2gh(\cos\theta - \cos\phi)/k_0^2 \quad . \quad . \quad . \quad (3)$$

Similarly, if a particle (Fig. 101) be attached to a point O′ by a flexible thread of length l, and be released from a position B′ such that $B'\widehat{O}'A = \phi$, O′A being vertical, its velocity v when passing G′, such that $G'\widehat{O}'A = \theta$, is given by—

$$v^2 = 2g \,.\, M'N' = 2gl(\cos\theta - \cos\phi)$$

and its angular velocity ω about O′ being v/l

$$\omega^2 = 2g(\cos\theta - \cos\phi)/l \quad . \quad (4)$$

or

$$v^2 = 2gl(\cos\theta - \cos\phi) \quad . \quad (5)$$

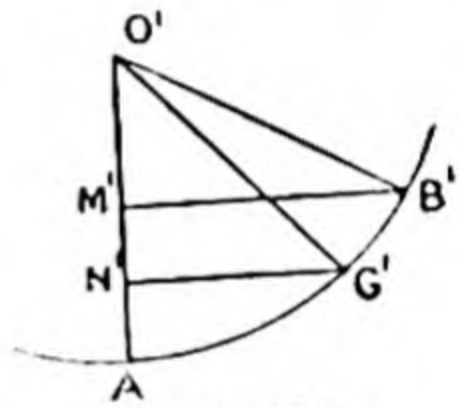

FIG. 101.

The angular velocity of a particle (or of a simple pendulum) given by equation (4) is the same as that of G (Fig. 100) given by equation (3), provided—

$$g/l = gh/k_0^2$$

i.e. provided

$$l = k_0^2/h \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This length k_0^2/h is then the length of the simple pendulum equivalent to that in Fig. 100, for since the velocity is the same at *any* angular position for the simple pendulum of length l and the actual pendulum, their times of oscillation must be the same. And from equation (3) of the Appendix,

$$k_0^2 = k_G^2 + h^2$$

hence, from (6)

$$l = k_0^2/h = h + k_G^2/h \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The point C (Fig. 100), distant $h + k_G^2/h$ from O, and in the line OG is called the "*centre of oscillation.*" The expression $h + k_G^2/h$ shows that it is at a distance k_G^2/h beyond G from O. A particle placed at C would oscillate

in the same period about O as does the *compound pendulum* of Fig. 100.

EXAMPLE 1.—Find the length of a clock pendulum which will make three beats per second. If the clock loses 1 second per hour, what change is required in the length of pendulum?

Let l = length of pendulum in feet.

Time of vibration $= \frac{1}{3}$ second

$$l = \frac{(\frac{1}{3})^2 \times 32{\cdot}2}{4\pi^2} = \frac{32{\cdot}2}{36\pi^2} \text{ feet} = 1{\cdot}09 \text{ inches}$$

The clock loses 1 second in 3600 seconds, *i.e.* it makes 3599×3 beats instead of 3600×3. Since $l \propto t^2 \propto 1/n^2$, where n = number of beats per hour, therefore

$$\frac{\text{correct length}}{1{\cdot}09 \text{ inches}} = \frac{3599^2}{3600^2} = (1 - \tfrac{1}{3600})^2$$

$$= 1 - \tfrac{1}{1800} \text{ approximately}$$

therefore *shortening* required

$$= 1{\cdot}09/1800 \text{ inches} = 0{\cdot}000606 \text{ inch}$$

EXAMPLE 2. A flywheel having a radius of gyration of 3·25 feet is balanced upon a knife-edge parallel to the axis of the wheel and inside the rim at a distance of 3 feet from the axis of the wheel. If the wheel is slightly displaced in its own plane, find its period of oscillation about the knife-edge.

The length of the simple equivalent pendulum is

$$3 + (3{\cdot}25)^2/3 = 3 + 3{\cdot}5208 = 6{\cdot}5208 \text{ feet.}$$

Hence the period is $2\pi\sqrt{6{\cdot}5208/32{\cdot}2} = 2{\cdot}83$ seconds

78. Unbalanced Circular Motion in Vertical Plane. In the preceding article (77), it was assumed that the angular amplitude of a pendulum was very small, and if this condition is fulfilled the motion is very nearly simple harmonic and the period independent of the amplitude. We now briefly consider the motion if the amplitude is not small. The motion may be oscillatory up to an amplitude of 180°. Or it may be continuous rotation in a constant direction of a pendulum bob and thread or of a rigid body but with

varying speed owing to the varying effect (torque) of gravity about the axis of rotation when the body (as in a pendulum) is not balanced about its axis of swing or rotation.

Consider first the motion of a suspended particle as in the (ideal) simple pendulum. Suppose it moves, say, counter-clockwise in a vertical circle with centre O, Fig. 102.

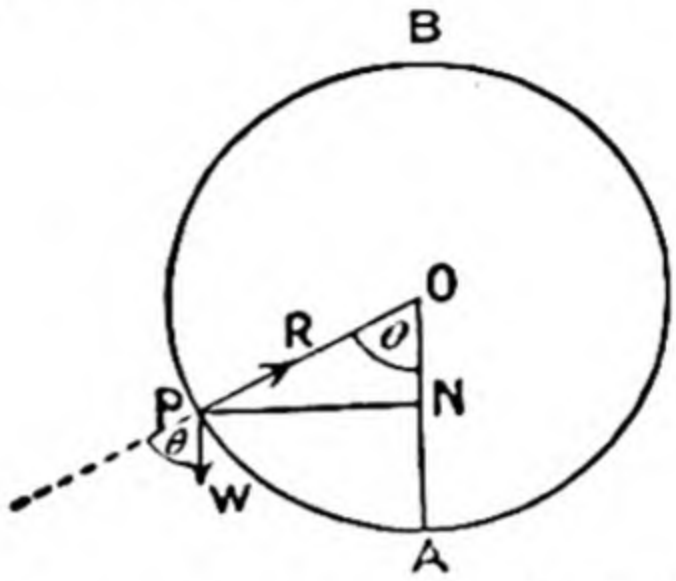

FIG. 102.—Motion in vertical circle.

It may be kept in the circular path by a string attached to O, or by an inward pressure of a circular track.

Let R = the radial inward force exerted on the particle by its constraints.

W = the weight of the particle in pounds ;

v = its velocity in feet per second in any position P such that OP makes an angle θ with the vertical OA, A being the lowest point on the circumference ;

v_A = the velocity at A ;

h = the radius of the circle in feet.

Then the kinetic energy at A is $\frac{1}{2}Wv_A^2/g$

At P the increase of potential energy is $W \times AN$, and the kinetic energy is $\frac{1}{2}Wv^2/g$, and since there is no work done or lost between A and P, the total mechanical energy at P is equal to that at A,

$$\frac{1}{2}Wv^2/g + W \times AN = \frac{1}{2}Wv_A^2/g \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$v^2 = v_A^2 - 2g \times AN \quad . \quad . \quad . \quad . \quad (2)$$

$$v^2 = v_A^2 - 2gh(1 - \cos\theta) \quad . \quad . \quad . \quad (3)$$

If the amplitude of oscillation is ϕ, $v=0$ when $\theta=\phi$

$$0=v_A^2-2gh(1-\cos\phi) \quad . \quad . \quad . \quad (4)$$

$$v_A^2=2gh(1-\cos\phi) \quad . \quad . \quad . \quad . \quad (5)$$

And

$$v^2=2gh(\cos\theta-\cos\phi) \quad . \quad . \quad . \quad (6)$$

the length $h(\cos\theta-\cos\phi)$ is the vertical distance of P below the position of the particle at the top of its swing. In order to move in a circular path of radius h at a speed v, the particle must have exerted upon it a radial force $Wv^2/(gh)$ towards O, in addition to which the axis has to support radially the gravitational (component) force $W\cos\theta$, hence

$$R=Wv^2/gh+W\cos\theta=W(3\cos\theta-2\cos\phi) \quad . \quad (7)$$

The value of R at any given point can be found by substituting for v from equation (3), provided v_A is known. The least value of R will be at B, the highest point of the circle, where gravity diminishes it most. If v_A is not sufficient to make R greater than zero for position B, the particle will not describe a complete circle. Examining such a case, the condition, in order that a complete revolution may be made without change in the sign of R, is that R at B shall be greater than zero, and since

$$R_B=Wv_B^2/(gh)+W\cos 180^\circ=W(v_B^2/gh-1) \quad . \quad (8)$$

the condition is that v_B^2 shall be greater than gh. And from (3), at $\theta=180^\circ$

$$v_B=v_A^2-2gh\times 2=v_A^2-4gh \quad . \quad . \quad . \quad (9)$$

Hence the condition is that v_A^2-4gh shall be greater than gh or v_A^2 greater than $5gh$ or $v_A>\sqrt{5gh}$, *i.e.* the velocity at A must be greater than that due to falling through a height $\frac{5}{2}h$, for which the velocity would be $\sqrt{5gh}$. For example, in a centrifugal railway (" looping the loop ") the necessary velocity on entering the track at the lowest point, making no allowance for frictional resistances, may be obtained by running down an incline of height greater than two and a half times the radius of the circular track.

The pendulum bob, suspended by a thread, is of course

limited to oscillation of less than a semicircle or to complete circles.

Proceeding to the rigid body, equation (5), Art. 77, gives the angular speed of the body (and therefore the linear speed of all points in it) at any position defined by θ, and from the speed, the time for a complete oscillation may be calculated, but for finite amplitude angles ϕ, the calculation is not simple; the integration involving either an approximation by expansion in a series or tabulation of small parts, the time for a small angle being the angle divided by the value of ω for the middle of the angle. This is not of importance in applied mechanics and need not be investigated here.

If the body makes complete revolutions, its speed will still vary because of the variation of the gravitational forces about the axis of rotation during a single revolution. If ω_B is the angular speed at the highest point B (see Figs. 102 and 100) and ω_A that at the lowest point A, the equation of energy (2), Art. 77, becomes

$$\tfrac{1}{2}Wk_0^2\omega^2/g = \tfrac{1}{2}Wk_0^2\omega_B^2/g + Wh(1+\cos\theta) \quad (10)$$

$$= \tfrac{1}{2}Wk_0^2\omega_A^2/g - Wh(1-\cos\theta) \quad (11)$$

and

$$\tfrac{1}{2}Wk_0^2(\omega_A^2-\omega_B^2)/g = 2Wh \quad . \quad . \quad . \quad (12)$$

In order that the body shall just make a complete revolution, *i.e.* $\omega_B=0$,

$$\tfrac{1}{2}Wk_0^2\omega_A^2/g = Wh \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\omega_A^2 = 4gh/k_0^2 \quad \text{or} \quad v_A^2 = 4gh \times h^2/k_0^2 \quad . \quad . \quad (14)$$

$$v_A = \sqrt{4gh}\,.\,h/k_0 \quad . \quad . \quad (15)$$

e.g. for a uniform rod suspended from one end, $h=$length/2, $k_0^2=$(length)$^2/3$, $h^2/k_0^2=3/4$, hence $v_A=\sqrt{3gh}$; this is the maximum speed of the c.g. and half that of the free end.

The radially inward force R exerted on the body by the axis O, Figs. 100 and 102, is Wv^2/gh or $W\omega^2h/g$, due to rotation at the angular speed ω and W cos θ in support of the gravitational pull W, so that

$$R = W\omega^2h/g + W\cos\theta \quad . \quad . \quad . \quad (16)$$

as in (7), and, if the body is oscillating with angular amplitude ϕ, inserting from (3), Art. 77, the value of ω^2

$$R = 2W(\cos\theta - \cos\phi)h^2/k^2 + W\cos\theta \quad . \quad (17)$$

which agrees with (7) when $k_0 = h$ (for a particle).

If the body just completes a circuit so that $\phi = 180°$, $\cos\phi = -1$, and

$$R = W\{\cos\theta + 2(\cos\theta + 1)h/k_0^2\} \quad . \quad . \quad (18)$$

If the body is a uniform bar oscillating about an axis perpendicular to its own, at one end, $h^2/k^2 = 3/4$ as before, and

$$R = \tfrac{1}{2}W(5\cos\theta + 3) \quad . \quad . \quad . \quad . \quad (19)$$

R reaches zero and changes sign and becomes a radially outward force when $5\cos\theta + 3 = 0$, *i.e.* $\cos\theta = -0{\cdot}6$ or $\theta = 126{\cdot}9°$ (about). (R is here only the *radial component* force exerted by the axis on the body. There is another component perpendicular to it which, with the component W sin θ of W perpendicular to OG, will suffice to give G its linear and the rod its angular acceleration).

EXAMPLE 1. A stone weighing W lb. is whirling in a vertical circle at the extremity of a string 3 feet long. Find the velocity of the stone and tension of the string—(1) at the highest position, (2) at lowest, (3) midway between, if the velocity is the least possible for a complete circle to be described.

If the velocity is the least possible, the string will just be slack when the stone is at the highest point of the circle.

Let v_B be the velocity at the highest point, where the weight just supplies the centripetal force; using Fig. 102,

(1) $W \times v_B^2/(3 \times 32{\cdot}2) = W$

$$v_B^2 = 3 \times 32{\cdot}2 \text{ ft.}^2/\text{sec.}^2, \quad v_B = 9{\cdot}83 \text{ ft./sec.}$$

(2) At the lowest point, since the total energy remains constant,

$$\tfrac{1}{2}Wv_A^2/32{\cdot}2 = \tfrac{1}{2}Wv_B^2/32{\cdot}2 + W \times 6$$

$$v_A^2 = v_B^2 + 2 \times 32{\cdot}2 \times 6 = 96{\cdot}6 + 386{\cdot}4$$

$$v_A = \sqrt{483} = 22 \text{ ft./sec. nearly.}$$

And the tension $= W + W \times 483/(32{\cdot}2 \times 3) = W + 5W = 6W$.

(3) When the string is horizontal, and the speed is v_0, similarly

$$\tfrac{1}{2}Wv_0^2/32{\cdot}2 = \tfrac{1}{2}Wv_B^2/32{\cdot}2 + W \times 3$$

$$v_0^2 = v_B^2 + 2 \times 3 \times 32{\cdot}2 = 96{\cdot}6 + 193{\cdot}2 = 289{\cdot}8 \text{ (ft./sec.)}^2$$

$$v_0 = \sqrt{289{\cdot}8} = 17 \text{ ft./sec.}$$

And the tension $= W \times 289{\cdot}8/(32{\cdot}2 \times 3) = 3W$.

EXAMPLE 2. A uniform rod 4 ft. long, weighing W lb., turns in a vertical circle about an axis through one end and perpendicular to the centre line of the rod. If its speed when in a horizontal position is 60 r.p.m. (or 2π radians per sec.), find its speed when vertically (1) above, (2) below the axis of rotation, and find the force exerted on and by the axle in these positions. Estimate the average angular velocity and time taken in one revolution.

Using the notation of Fig. 102, the angular velocity when the rod is at the level O is $\omega_0 = 2\pi = 6{\cdot}283$ radians/sec. and $h = 2$ ft., $k_0^2 = 4^2/3 = 16/3$ ft.^{2}

From (11) when $\theta = 90°$,

since $2gh/k_0^2 = 2 \times 32{\cdot}2$ ft./sec.$^2 \times 2$ ft./(16/3) ft.^{2}

$$= 24{\cdot}15 \text{ (1/sec.}^2\text{)}$$

$$\omega_A^2 = \omega_0^2 + 2gh/k_0^2 = 4\pi^2 + 24{\cdot}15 = 63{\cdot}63 \text{ (1/sec.}^2\text{)}$$

$$\omega_A = 7{\cdot}977 \text{ rad./sec.}$$

And for any angle θ (Fig. 102),

$$\omega^2 = \omega_A^2 - (2gh/k_0^2)(1 - \cos\theta)$$

$$= \omega_A^2\{1 - (24{\cdot}15/63{\cdot}63)(1 - \cos\theta)\}$$

$$\omega = \omega_A\sqrt{1 - 0{\cdot}3796(1 - \cos\theta)}$$

$$= 4{\cdot}915\sqrt{1{\cdot}635 + \cos\theta} \text{ rad./sec.}$$

For $\theta = 180°$, putting $\cos\theta = -1$, gives

$$\omega_B = 3{\cdot}916 \text{ radians/sec.}$$

$\omega_0 = 2\pi = 6{\cdot}283$ or $\frac{1}{2}(\omega_A + \omega_B) = 5{\cdot}946$ radians/sec. might be considered mean values of ω. But the true mean value is the *time-average* of the values of the angular velocity or

2π divided by the time taken for one revolution. The integration of $dt/d\theta$, the reciprocal of $d\theta/dt(=\omega)$, involves special functions or integration of a series and is not simple. But we can find the time of a revolution by dividing, say, a half revolution on either side of the vertical diameter of the circle (Fig. 102) into intervals of 20°, *i.e.* $\pi/9$ radians, and calculating the time δt taken in traversing each 20° from the angular speed at the middle of the interval, *i.e.* at values of θ of 10, 30, 50, 70, 90, 110, 130, 150, and 170 degrees. The sum of the time δt, say $\Sigma(\delta t)$, gives the time for a half revolution and the mean angular velocity is $\pi/\Sigma(\delta t)$. The results would naturally be tabulated to facilitate calculation and need not be reproduced in detail here. The time taken for the half revolution is $\Sigma(\delta t)$ $=0{\cdot}54577$ sec. or $1{\cdot}0915$ sec. per rev., and the mean angular velocity is $\pi/0{\cdot}54577=1{\cdot}832\pi$ or $5{\cdot}757$ radians/sec., against the values of 2π or $6{\cdot}283$ radians/sec. for ω_0 and $5{\cdot}946$ radians/sec. for $\frac{1}{2}(\omega_A+\omega_B)$. A little consideration, particularly with a sketch of the graph of ω or of $1/\omega$ on a base of values of θ, will show that these values are too high. The mean height of such a graph would represent an *angle-average* and not a time average, and each equal part of the circuit would contribute one of 9 ordinates to the sum on which the average is based, even though it correspond to less (or more) than 1/9 of the time of the half revolution. But in the half circuit above the level O, the speeds being slower, the time taken for any given angular travel is greater than in the lower half of the circuit, and the slower speeds are more potent in their contributions to the total on which true time-average is based, and the time-average is a smaller quantity than the angle-average. Thus if we average the values for each 20°, we get an angle-average value of $6{\cdot}117$ radians/sec., which is too high. The value $5{\cdot}757$ radians/sec. is, of course, an approximation and might be improved by dividing the half circuit into smaller intervals than 20°. (But as the graph of $1/\omega$ has reversed curvature on opposite sides of $\theta=90°$, the larger intervals are justified and give a good result, confirmed by integration.) The result will suffice to illustrate the difference between a time and a space or angle average.[1]

[1] As a very simple illustration of this point, if one mile is covered at 60 m.p.h. and one mile at 20 m.p.h., the space average

The equal and opposite forces exerted on the rod and axle are from (16), when $\theta=0°$, $\omega^2=\omega_A{}^2=63\cdot63$ (1/sec.^{2}),

$$R_A=W(1+63\cdot63\times2/32\cdot2)=4\cdot95W$$

and when $\theta=180°$,

$$R\ =W(15\cdot33\times2/32\cdot2-1)=W(0\cdot9521-1)=-0\cdot05W$$

i.e. an upward thrust of the axle on the rod and a downward pressure of the rod on the axle of 0·05W (R changes sign (R=0) when $\cos\theta=-0\cdot9521$, *i.e.* when $\theta=180°-17\cdot8°=162\cdot2°$, or 17·8° before the rod reaches the vertical position at B). Note that the difference $\{4\cdot95-(-0\cdot05)\}W=5\cdot0W$ is much greater than 2W, the difference due to the gravitational force of the rod. The difference due to difference of speed is $W(\omega_A{}^2-\omega_B{}^2)\times2/32\cdot2=3\cdot0W$. Also note that in the vertical positions of the rod there is no angular acceleration; the forces R_A and R_B are not merely radial components but the resultant forces between the axle and the rod.

EXAMPLES VII

1. A point has a simple harmonic motion of amplitude 6 inches and period 1·5 seconds. Find its velocities and accelerations 0·1 second, 0·2 second, and 0·5 second after it has left one extremity of its path.

2. A weight of 10 lb. hangs on a spring, which stretches 0·15 inch per pound of load. It is set in vibration, and its greatest acceleration whilst in motion is 16·1 feet per second per second. What is the amplitude of vibration?

3. A point, A, in a machine describes a vertical circle of 3 feet diameter, making 90 circuits per minute. A portion of the machine weighing 400 lb. moves in a horizontal straight line, and is always a fixed distance horizontally from A, so that it has a stroke of 3 feet. Find the accelerating forces on this portion, (1) at the end of its stroke; (2) 9 inches from the end; and (3) 0·05 second after it has left the end of its stroke

4. A helical spring deflects $\frac{1}{8}$ of an inch per pound of load. How many vibrations per minute will it make if set in oscillation when carrying a load of 12 lb.?

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of speed is $\frac{1}{2}(60+20)=40$ m.p.h., whereas the true or time-average speed is obtained from the fact that one mile at 60 m.p.h. takes 1 minute and one mile at 20 m.p.h. takes 3 minutes, and therefore 2 miles take altogether 4 minutes, *i.e.* a time-average speed of 30 m.p.h., which is 25 per cent. less than the space average of 40 m.p.h.

5. A weight of 20 lb. has a simple harmonic vibration, the period of which is 2 seconds and the amplitude 1·5 feet. Draw diagrams to stated scales showing (1) the accelerating force exerted on the weight at all points in its path ; (2) the displacement at all *times* during the period ; (3) the velocity at all times during the period ; (4) the force acting at all times during the period ; (5) the K.E. at all points in its path.

6. A light stiff beam deflects 1·145 inches under a load of 1 ton at the middle of the span. Find the period of vibration of the beam when so loaded.

7. A point moves with simple harmonic motion ; when 0·75 foot from mid-path, its velocity is 11 feet per second ; and when 2 feet from the centre of its path, its velocity is 3 feet per second. Find its period and its greatest acceleration.

8. Use the data in Example 2, Art. 75. When the crank has moved 135° from the dead centre in which the piston is farthest from the crankshaft, find (1) how far the piston has moved, (2) its speed, (3) its acceleration, and (4) the force required to decelerate the reciprocating parts.

9. If, in No. 8, the effective pressure is 40 lb. per sq. in. on a piston 6 inches diameter, find the torque on the crankshaft in the position stated.

10. A piece of metal weighing 10 lb. is suspended by a vertical wire which passes through the centre of gravity of the metal. A twist of 8·5° is produced per pound-foot of twisting moment applied to the wire, and when the metal is released after giving it a small twist, it makes 150 complete oscillations a minute. Find the radius of gyration of the piece of metal in inches.

11. A flywheel weighing 3 tons is fastened to one end of a shaft, the other end of which is fixed, and the torsional rigidity of which is such that it twists 0·4° per ton-foot of twisting moment applied to the flywheel. If the radius of gyration of the flywheel and shaft combined is 3 feet, find the number of torsional vibrations per minute which the wheel would make if slightly twisted and then released.

12. How many complete oscillations per minute will be made by a pendulum 3 feet long ? $g = 32{\cdot}2$ ft./sec^2.

13. A connecting rod makes 29 small oscillations per minute about a knife edge placed through the hole at the big end. The c.g. of the rod is 1·5 feet from the knife edge. Find the radius of gyration of the rod about an axis through the c.g. and parallel to the knife edge. If the rod is allowed to oscillate about a knife edge placed through the hole at the small end 5 feet from the previous line of contact, how many oscillations per minute will it make ?

14. A body weighing $\frac{1}{2}$ lb., attached to a string, is moving in a vertical circle of 6 feet diameter. If its velocity, when passing through the lowest point, is 40 feet per second, find its velocity

and the tension of the string when it is 2 feet and when it is 5 feet above the lowest point.

15. A small bob or piece of metal of weight W attached to one end of a thread 4 feet long, moves in a vertical circle about the other end of the thread as centre. Its angular speed is 60 revolutions per minute when the thread is horizontal. Find its speed and the tension of the thread when it is vertical (1) below the centre of motion, (2) above the centre.

16. Estimate the time of a circuit in No. 15. (See Example 2 at end of Art. 78 and take $h = 4$ ft, $k_0 = 4$ ft.)

CHAPTER VIII

HYDROSTATICS

79. Mechanics of Liquids. The applied mechanics of liquids can conveniently be divided into hydrostatics, dealing with liquids at rest under the action of balanced forces, and hydraulics dealing with liquids in motion. In order to make the mechanics of liquids included in this book self-contained, parts of earlier courses will be repeated, though it may not be necessary for every reader to study these. The present chapter is devoted to hydrostatics and the following one to hydraulics.

Liquids and Fluid Pressure. The molecules of a liquid, unlike those of a solid, although they are in contact one with another, move freely one over another. Liquids have some resistance, called viscosity, to rapid change of shape, this being a tangential one opposing sliding motion ; it is proportional to the speed of sliding motion of one molecule over another. At low speeds it is negligible, and in a liquid at rest it is zero. That is, the force between two molecules of a liquid at rest is entirely perpendicular to their surfaces in contact or wholly normal. And, for such liquids as we need consider, the force transmitted between a liquid and a solid surface can be regarded as a normal pressure. This is true of water at rest and very nearly true so long as it moves slowly.

The intensity of pressure exerted within a fluid or on the walls of a containing vessel is measured by the force per unit of area, generally pounds per square inch or per square foot. This is very commonly called merely the *pressure*, although it is an intensity or degree of pressure and not the total force exerted which is also called the pressure, but to avoid confusion the latter is sometimes called the total pressure. From the fact that only normal pressure can be transmitted by a stationary fluid it follows that this normal pressure is equal in all directions at a given point in a fluid. And this point can be demonstrated experimentally by the use of pressure gauges.

80. Relation of Pressure to Depth in a Liquid. In a liquid, large intensities of pressure may arise from the weight of the liquid when it extends continuously to a great depth. To investigate how much such pressure amounts to, imagine a vertical cylindrical shaped portion of a liquid at rest in a tank (Fig. 103), then, since the pressure on the curved vertical surface of the cylinder is everywhere horizontal, it will have no vertical effect, and difference of the vertical pressure on the two flat ends must balance the weight of the cylindrical column of water. Putting this in the form of symbols, let A be the area of cross-

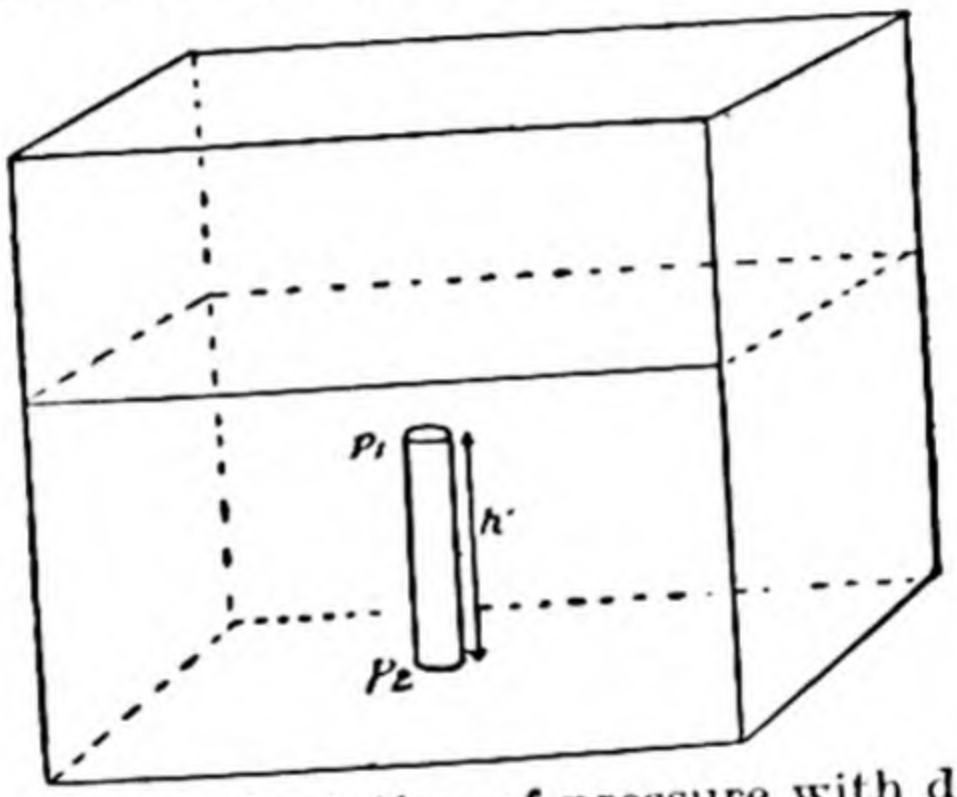

FIG. 103.—Variation of pressure with depth.

section of the imagined cylinder, in square feet, h its height in feet, p_1 and p_2 the intensities of pressure at the top and bottom ends respectively in pounds-force per square foot. Then the volume of liquid in the cylinder is $A \times h$ cu. ft., and if the liquid weighs w lb. per cu. ft., the total weight of the cylinder of water is

$$A \times h \text{ cu. ft.} \times w \text{ lb./cu. ft.}$$

or

$$A \,.\, h \,.\, w \text{ lb.}$$

and this must equal the net external force exerted upward on the cylinder which is

$$A \text{ sq. ft.} \times p_2 \text{ lb.-force/sq. ft. (upward)}$$

$$-A \text{ sq. ft.} \times p_1 \text{ lb.-force/sq. ft. (downward)}$$

$$= A(p_2 - p_1) \text{ lb.-force (upward).}$$

Hence, since this balances the downward force,

$$A(p_2-p_1) \text{ lb.} = A \,.\, h \,.\, w \text{ lb.}$$

or
$$p_2-p_1 = wh \text{ lb. per sq. ft.} \qquad (1)$$

That is, the increase of pressure for a depth h ft. is wh lb. per sq. ft. or w lb. per sq. ft. per ft. of increase of depth.

If one end of the imaginary cylinder is in the free surface of the water, the upward pressure of the surrounding water on the base of the cylinder is just equal to the weight of the cylinder of water.

Or, reckoning from the surface of the liquid, the pressure intensity at a depth h ft. will exceed the atmospheric pressure at this surface by

$$wh \text{ lb./sq. ft.}$$

And for pressures at great depths, the atmospheric pressure, equivalent to a depth of 34 ft. of water (see example below), becomes negligible.

The density of water (or its mass per unit volume) is about 62·4 lb. per cu. ft. at 60° F., so that for water, $w = 62{\cdot}4$ lb.-force per cu. ft., and at a depth of 1 ft., the intensity of pressure (reckoned above atmospheric) is

$$p = 62{\cdot}4 \text{ lb./sq. ft.}$$

$$= \frac{62{\cdot}4}{144} \text{ or } 0{\cdot}433 \text{ lb./sq. in.}$$

and at a depth of h ft.,

$$\text{pressure intensity} = 0{\cdot}433 \text{ lb./sq. in. per ft. depth} \times h \text{ ft.}$$

$$= 0{\cdot}433\, h \text{ lb./sq. in. (above the pressure of the atmosphere).}$$

Thus at a depth of 100 ft., the pressure would be $0{\cdot}433 \times 100 = 43{\cdot}3$ lb. per sq. in. above atmospheric pressure.

And to give a pressure of 1 lb. per sq. in., a depth or "head" of $1/0{\cdot}433$ or 2·31 ft. of water would be required, or $2{\cdot}31p$ ft. for p lb. per sq. in.

EXAMPLE. *Water Barometer.* What head of water would give a pressure equal to that of the atmosphere, namely, 14·7 lb. per sq. in. ?

14·7 lb. per sq. in. $=14·7\times144$ lb. per sq. ft. $=2117$ lb. per sq. ft. 1 cu. ft. of water weighs 62·4 lb. and a column h ft. high and 1 ft. sq. in section would weigh $62·4h$ lb., which would give a pressure of 2117 lb. on the base, provided that

$$62·4h=2117$$

$$h=\frac{2117}{62·4}=34 \text{ ft. nearly.}$$

This is sometimes called the height of the "water-barometer.

Mercury Barometer. The atmospheric pressure corresponds to that of a column of mercury of which the height will be less than that of the water barometer in the ratio that mercury is heavier than water, viz. 13·6 Hence height of mercury column to give a pressure of 14·7 lb. per sq. in. is about 34 ft./13·6 $=2·5$ ft. $=30$ in.

81. Pressure on Submerged Surfaces. Since the pressure per square foot at a depth h in a liquid which weighs w lb. per cu. ft. is wh lb. per sq. ft. (above atmospheric pressure), any horizontal surface such as the bottom of a tank of area A sq. ft. has upon it a total pressure of

$$wh \text{ lb./sq. ft.}\times\text{A sq. ft.}=wh\text{A lb. force} \quad . \quad (1)$$

But to find the total pressure on any plane surface which is not horizontal, we have to deal with a pressure which is continuously varying across the surface, according to the varying depth.

Take a vertical surface of total area A sq. ft. (shown in Fig. 104) and suppose it divided into a large number of parallel horizontal strips, the areas of which are a_1, a_2, a_3, etc., in square feet, and the depths of the strips are h_1, h_2, h_3, etc., ft. respectively below the surface of the liquid, then the total pressures on the successive strips will be

$$wh_1a_1,\ wh_2a_2,\ wh_3a_3, \text{ etc.,}$$

and the total pressure P, say, in pounds will be

$$\text{P}=wh_1a_1+wh_2a_2+wh_3a_3+ \ . \ . \ . \text{ etc.}$$
$$=w(h_1a_1+h_2a_2+h_3a_3+ \ . \ . \text{ etc.}) \quad . \quad (2)$$

Now the sum of the products h_1a_1, h_2a_2, etc., constitute the moment of the area A about an axis which is the line of intersection of the vertical plane of the submerged surface with the surface of the liquid, and this moment is such that

$$h_1a_1+h_2a_2+h_3a_3+\text{etc.}=\bar{h}\,.\,\text{A} \quad . \quad . \quad . \quad (3)$$

where $\bar{h}$ is the distance of the centroid, centre of area or so-called centre of gravity of the total area A from the same axis. Hence we may write equation (2) as

$$\text{P}=w\,.\,\bar{h}\,.\,\text{A lb.} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

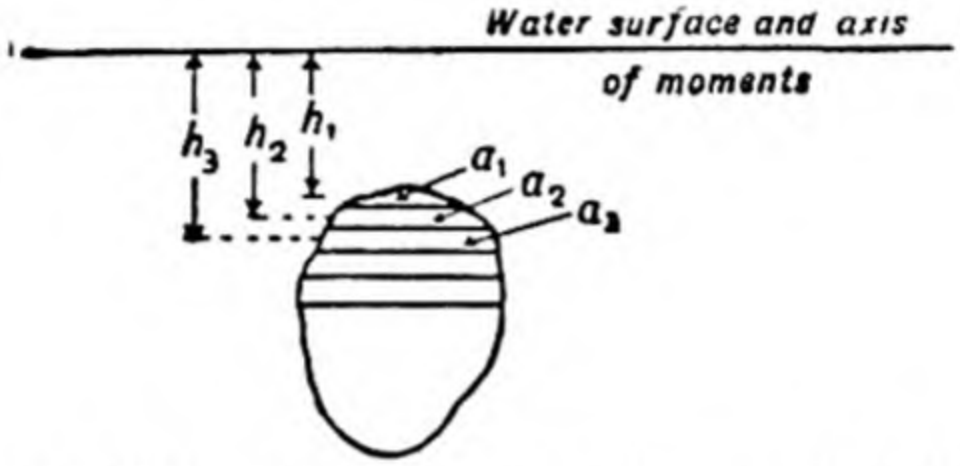

FIG. 104.—Pressure on submerged plate.

To put the matter in another way,

$$\text{P}=\bar{p}\text{A lb.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $\bar{p}$ (or $w\bar{h}$) is the mean pressure intensity or the intensity of pressure at the depth of the centre of gravity or centroid of the immersed surface. In words,

Total pressure in lb.

=area in sq. ft. × pressure per sq. ft. at c.g. of area.

If a surface is neither horizontal nor vertical but oblique at some intermediate inclination, the same rule applies, where $\bar{h}$ still refers to the *vertical* depth of the centre of gravity.

EXAMPLE 1. Find the total pressure on a lock gate the width of which is 20 ft., when the depth of fresh water is 15 ft.

Depth of c.g. of wetted area =15/2 ft. =7·5 ft.

Pressure at depth of c.g. $=7{\cdot}5$ ft. $\times 62{\cdot}4$ lb./cu. ft.
$=468$ lb./sq. ft.

Total wetted area $=20$ ft. $\times 15$ ft. $=300$ sq. ft.

Total pressure $=468$ lb./sq. ft. $\times 300$ sq. ft.
$=140{,}400$ lb.

EXAMPLE 2. A submerged rectangular sluice gate is 3 ft. by 2 ft., having its long sides vertical and short sides horizontal. The top side is 5 ft. below the surface of water. Find the total water pressure on the gate. Take salt water at 64 lb. per cu. ft.

The depth of the c.g. below the top side $=3/2$ ft. $=1{\cdot}5$ ft.

Depth of c.g. below surface of water $=5+1{\cdot}5=6{\cdot}5$ ft.

Pressure at depth of c.g. $=6{\cdot}5$ ft. $\times 64$ lb./cu. ft.
$=426$ lb./sq. ft.

Total area $=3$ ft. $\times 2$ ft. $=6$ sq. ft.

Total pressure $=426$ lb./sq. ft. $\times 6$ sq. ft. $=2556$ lb.

EXAMPLE 3. The tank shown in elevation in Fig. 105 is 6 ft. broad (in a direction perpendicular to the diagram).

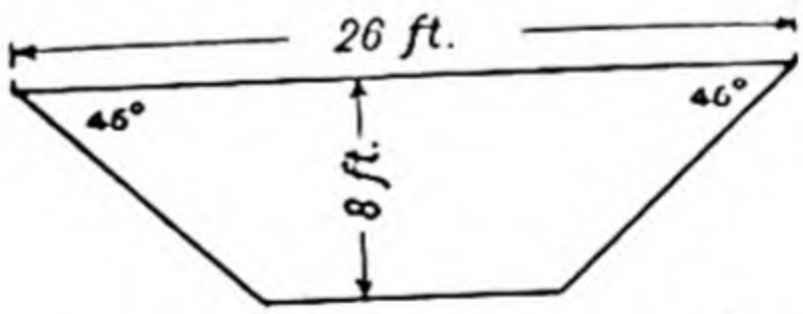

FIG. 105.—Problem on tank.

Find the pressure on (*a*) a sloping side, (*b*) a vertical side, and (*c*) on the bottom, when the tank is full of water.

(*a*) Length of sloping side $=8\sqrt{2}$ ft. $=11{\cdot}31$ ft.

Area of sloping side $=11{\cdot}31$ ft. $\times 6$ ft. $=67{\cdot}86$ sq. ft.

Depth of c.g. of sloping side $=8/2=4$ ft.

Pressure on sloping side $=67{\cdot}86$ sq. ft. $\times 4$ ft. $\times 62{\cdot}4$ lb./cu. ft.
$=16{,}960$ lb.

(*b*) We can divide the vertical side into a central rectangle 8 ft. deep and $26-2\times 8=10$ ft. long, and two right-angled
8

triangles 8 ft. deep and each having a base in the water surface 8 ft. long. The depth of the centres of gravity of these triangles will be 8/3 ft. below the water surface.

Rectangular area $=10$ ft. $\times 8$ ft. $=80$ sq. ft.

Depth of c.g. $=8/2$ ft. $=4$ ft.

Total pressure on this part $=80$ sq. ft. $\times 4$ ft. $\times 62{\cdot}4$ lb./cu. ft.
$=19{,}970$ lb.

Two triangular areas $=2 \times 8 \times 8 \times \frac{1}{2}=64$ sq. ft.

Depth of c.g. $=8/3$ ft. $=2{\cdot}67$ ft.

Total pressure on those parts
$=64$ sq. ft. $\times 2{\cdot}67$ ft. $\times 62{\cdot}4$ lb./cu. ft.
$=10{,}650$ lb.

Total pressure on vertical side $=30{,}620$ lb.

(*c*) Area of bottom $=10$ ft. $\times 6$ ft. $=60$ sq. ft.

Depth of c.g. $=8$ ft.

Total pressure $=60$ sq. ft. $\times 8$ ft. $\times 62{\cdot}4$ lb./cu. ft.
$=29{,}950$ lb.

82. Centre of Pressure. In the previous article we found the magnitude of the total distributed pressure on a vertical or an inclined immersed plane surface, but not the line of action of this resultant pressure. To determine this we find where the whole pressure, if concentrated, would have the same moment as the actual distributed pressure about the axis in which the plane of the surface intersects the free water surface, *i.e.* the axis of moments used in Art. 81 and shown in Fig. 104. Adopting the same symbols as in Art. 81, the forces on successive strips are

$$wh_1a_1,\ wh_2a_2,\ wh_3a_3, \text{ etc.}$$

and as their distances from the axis are, h_1, h_2, h_3, etc., respectively, the moments of these forces about the axis are

$$wh_1^2a,\ wh_2^2a_2,\ wh_3^2a_3 \ . \ . \ . \text{ etc.}$$

and the total moment

$$\Sigma(wah^2)=w(a_1h_1^2+a_2h_2^2+a_3h_3^2+ \ . \ . \ . \text{ etc.})$$
$$=w\,\Sigma(ah^2) \quad . \ . \ . \ . \ . \ . \quad (1)$$

It will be noticed that the sum $\Sigma(ah^2)$ is the so-called moment of inertia or second moment (*see* Art. 9) of the area A about the axis of moments in the water surface. If P, the total pressure, is to have the same moment about this axis as that of the distributed pressure shown in (1), and if the unknown distance of its line of action from the axis is H, then since from (4) of Art. 81, $P = w\bar{h}A$

$$w\bar{h}A \times H = w\,\Sigma(ah^2) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$H = \Sigma(ah^2)/\bar{h}A \quad . \quad . \quad . \quad . \quad (3)$$

And since $\Sigma(ah^2)$ may be written k_0^2A, where k_0 is the radius of gyration of the immersed area A about the axis of moments from which $\bar{h}$ also is measured,

$$H = k_0^2A/\bar{h}A = k_0^2/\bar{h} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which gives the depth of the centre of pressure below the water surface. If the radius of gyration about a (horizontal) parallel axis through the centroid or c.g. of the area A be k_G, then from equation (3) of the Appendix,

$$k_0^2 = k_G^2 + (\bar{h})^2 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

since $\bar{h}$ is the depth of the c.g., *i.e.* the distance apart of the two axes. Hence (4) may be written

$$H = \frac{k_G^2 + (\bar{h})^2}{\bar{h}} = \bar{h} + \frac{k_G^2}{\bar{h}} \quad . \quad . \quad . \quad . \quad (6)$$

or $k_G^2/\bar{h}$ is the distance of the centre of pressure below the centroid of the immersed area, *e.g.* for a rectangle of depth d with one side in the free surface, $\bar{h} = \frac{1}{2}d$ and $k_G^2 = d^2/12$ hence $H = \frac{2}{3}d$.

When the immersed area has a line of symmetry perpendicular to the axis of k_0 and k_G, the centre of pressure will lie in this line at the depth given by (6). Thus its position is completely determined.

EXAMPLE 1. Find the depth of the centre of pressure of the sluice gate in Ex. 2 of Art. 81, and the moment of the total pressure about hinges in the top side of the gate.

The square of the radius of gyration about an axis through the centroid of the gate area is

$$k_G^2 = \tfrac{1}{12} \times 3^2 \text{ ft. (see footnote, Art. 117).}$$

and since $\bar{h}$ is 6·5 ft., the distance of the c.p. below the c.g. is

$$\tfrac{1}{12} \times 3^2/6{\cdot}5 = 0{\cdot}1154 \text{ ft.}$$

and since the c.g. is 1·5 ft. below the hinge, the distance of the c.p. from the hinge is $1{\cdot}5 + 0{\cdot}1154 = 1{\cdot}6154$ ft. And the total pressure was found in Ex. 2 of Art. 81 to be 2556 lb., hence the total moment of the pressure about the hinge

$$= 2556 \text{ lb.} \times 1{\cdot}6154 \text{ ft.} = 4129 \text{ lb.-ft.}$$

EXAMPLE 2. Find the depth of the centre of pressure of the vertical trapezoidal side of the tank specified in Ex. 3 of Art. 81, given that the radius of gyration of a triangular area about its base is $1/\sqrt{6}$ of its vertical height.[1]

Using the results of Ex. 3, Art. 81, part (*b*), and that k^2 about the axis in the water surface is $\frac{1}{3}$ of 8^2 for the central rectangle, the depth of the c.p. below the water surface is by (4)

$$\frac{64}{3 \times 4} = \frac{16}{3} \text{ ft.}$$

And the moment of the pressure on this area about the axis is

$$19{,}970 \text{ lb.} \times 16/3 \text{ ft.} = 106{,}500 \text{ lb.-ft.}$$

Depth of c.p. of the triangular areas is from (4)

$$\frac{8^2}{6} \div \frac{8}{3} = 4 \text{ ft.}$$

[1] This may easily be found by integration. Let the base be B and the vertical height be D. A strip distant x from the base, and of width dx has a length $B \times (D - x)/D$ and area dx times this. Its moment of inertia about the base is $(D - x)x^2 dx \times B/D$

$$= \int_0^D x^2(D - x)dx \times B/D = D^4(\tfrac{1}{3} - \tfrac{1}{4})B/D = \tfrac{1}{12}BD^3$$

hence $k^2 = \frac{1}{6}D^2$ since $A = \frac{1}{2}BD$.

Moment of the pressure on two triangular areas is

$$10{,}650 \text{ lb.} \times 4 \text{ ft.} = 42{,}600 \text{ lb.-ft.}$$

Total moment of pressure on trapezium about top side is

$$106{,}500 + 42{,}600 = 149{,}100 \text{ lb.-ft.}$$

Dividing this by the total pressure of 29,950 lb. (Art. 81),

$$\text{Depth of c.p. of trapezium} = \frac{149{,}100}{29{,}950} = 4{\cdot}978 \text{ ft.}$$

EXAMPLE 3. The tank shown in end view in Fig. 106 has rectangular vertical sides and a horizontal semi-cylindrical

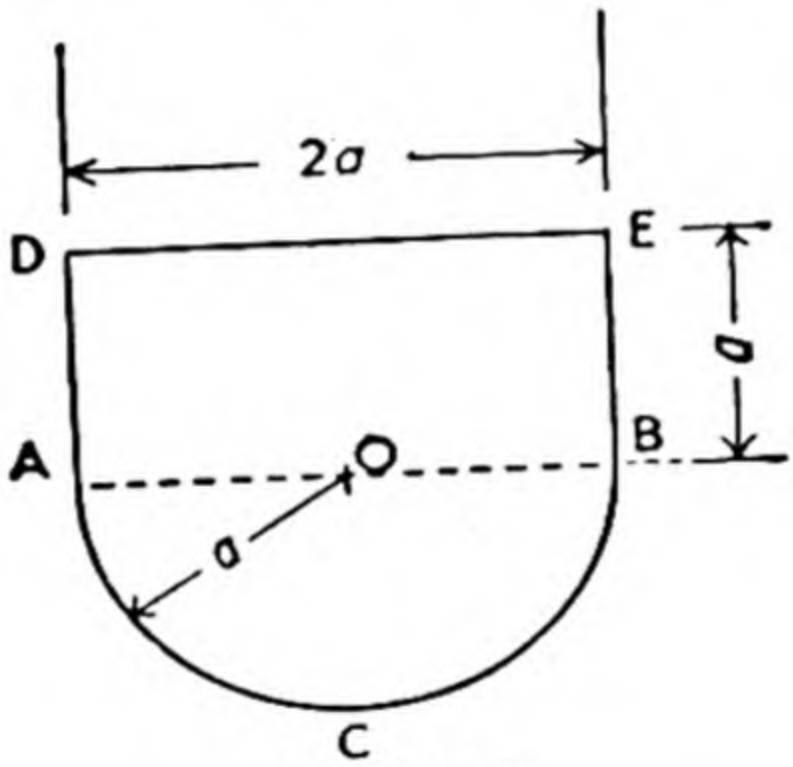

FIG. 106.

bottom of radius a, with flat vertical ends DEBCA. Find the resultant pressure on the end and the centre of pressure (1) when liquid of density w is in the tank to the level of the diameter AB of the semicircle, and (2) when the tank is full.

(1) The depth of the c.g. of semicircle ABC below AB is $4a/3\pi$ or $0{\cdot}4244a$. Then from (4), Art. 81,

Total pressure on ABC $= \frac{1}{2}\pi a^2 \times w \times 4a/3\pi = \frac{2}{3}a^3 w$ or $0{\cdot}6667a^3 w$.

k^2 about AB (as for a circle about a diameter), see (10) of Appendix, is $a^2/4$, hence from (4), Art. 82,

Depth of c.p. below AB $= \frac{1}{4}a^2/(4a/3\pi) = 3\pi a/16$ or $0{\cdot}5890a$.

(2) Total pressure on rectangle DEBA $=2a^2 \times \frac{1}{2}aw = a^3w$.

Depth of c.p. of this $=\frac{1}{3}a^2/\frac{1}{2}a = \frac{2}{3}a$.

Moment of this pressure about DE $=a^3w \times \frac{2}{3}a = \frac{2}{3}a^4w$ or $0{\cdot}6667a^4w$.

Depth of c.g. of semicircle ABC below DE $=1{\cdot}4244a$.

Total pressure on ABC $=\frac{1}{2}\pi a^2 \times 1{\cdot}4244aw = 2{\cdot}2375a^3w$.

About horizontal axis through c.g. (see equation (3) of Appendix)

$$k_G^2 = \tfrac{1}{4}a^2 - (0{\cdot}4244a)^2 = 0{\cdot}0699a^2.$$

Depth of c.p. below c.g., by (6) of Art. 82,

$$= \frac{0{\cdot}0699a^2}{1{\cdot}4244a} = 0{\cdot}0491a$$

" " " DE $=1{\cdot}4244a + 0{\cdot}0491a = 1{\cdot}4735a$.

Moment of pressure on ABC about DE

$$= 2{\cdot}2375a^3w \times 1{\cdot}4735a = 3{\cdot}2969a^4w.$$

Total moment about DE $=(3{\cdot}2969 + 0{\cdot}6667)a^4w$

$$= 3{\cdot}9636a^4w.$$

Total pressure on end $=2{\cdot}2375a^3w + a^3w = 3{\cdot}2375a^3w$.

Depth of c.p. of end $= \dfrac{3{\cdot}9636a^4w}{3{\cdot}2375a^3w} = 1{\cdot}2244a$.

(Four places of decimals are used to avoid large errors in the relatively small differences from subtractions.)

83. Density and Specific Gravity. The *density* of a homogeneous substance is its mass per unit volume for which the standard symbol is ρ so that

$$\rho = \frac{\text{mass}}{\text{volume}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Density is generally expressed in lb. per cu. ft. or in grammes per cubic centimetre. If a body is not homogeneous, (1) gives the average density.

The *specific gravity* S of a homogeneous substance is the ratio of the weight of any volume of that substance to the weight of an equal volume of a standard substance. (If

mass is substituted for weight, the ratio is unaltered.) The standard substance for solids and liquids is water at 4° C. (its temperature of maximum density). It is evident that the specific gravity of a substance is also the ratio of the density of the substance to that of the standard substance (both being in the same units).

The density of water at 4° C. is about 62·4 lb. per cu. ft. or 1 gm. per cu. cm. If the density of a particular steel be 480 lb. per cu. ft., its specific gravity would be 480/62·4 =7·8.

84. Flotation and Buoyancy. If a solid body is wholly or partially immersed in a liquid, it experiences a degree of support from the upward pressure of the liquid in contact

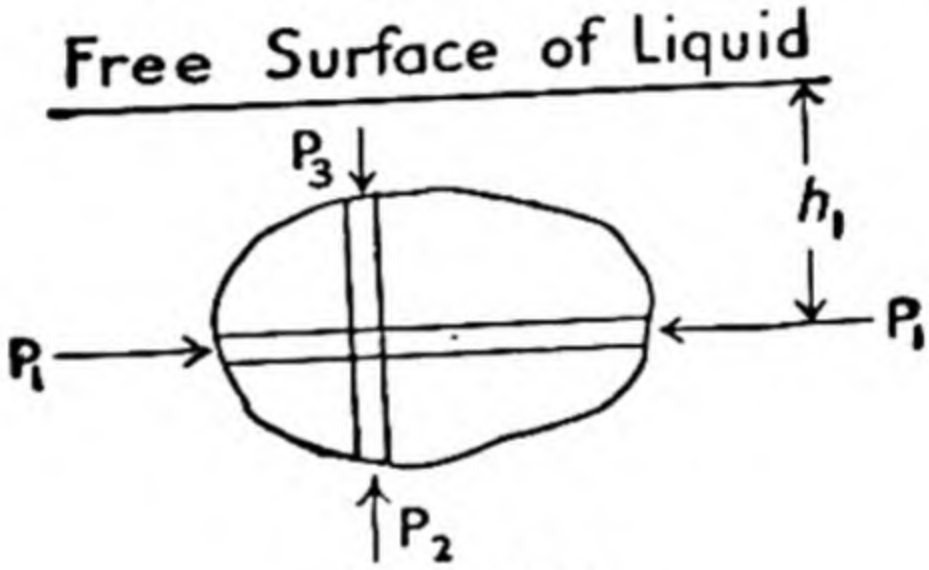

FIG. 107.

with it. If we consider any small portion of the immersed surface of the body, Fig. 107, and imagine the surface projected in any horizontal direction through to the opposite side of the body, a thin prism is outlined, the cross-section of which is equal to the projected area, say, a, of the chosen small area of surface of the body. If this area is at a depth h_1, the horizontal force on each end of this prism will be $P_1 = wah_1$, and thus the horizontal forces on this portion of the body will be balanced. This is true also for any and every horizontal direction and also for every element of surface and so the resultant horizontal force exerted on the body by the liquid is zero.

We next conceive the whole solid to be divided into very thin vertical prisms the cross-sections of which are the projected areas of small elements of surface of the solid. Let a be the area of cross-section of such a prism and let the

lower end of it in the surface of the solid be at a depth h_2 below the free surface of the liquid. Then the upward force of the liquid on the prism is $P_2 = awh_2$ which is equal to the weight of a volume of liquid of section a and length h_2, *i.e.* from the lower surface of the solid up to the free water surface. If the prism of the solid does not extend to the free water surface but terminates at a depth h_3 there is on its upper end a downward pressure $P_3 = awh_3$. The net or resultant upward pressure is then

$$(h_2 - h_3)w \times a$$

h_3 being zero if the prism ends at the free surface of the liquid or above it. And summing this for the whole body, the total upward pressure

$$= P = \Sigma\{(h_2 - h_3)aw\} \text{ or } w\Sigma\{(h_2 - h_3)a\} \quad . \quad (1)$$

The sum of all such small quantities as $(h_2 - h_3)a$ is

$$\Sigma\{(h_2 - h_3)a\} = V_1 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where V_1 is the volume of the part or whole of the solid which is below the free surface of the liquid, and $\Sigma\{(h_2 - h_3)aw\}$ is the weight of an equal volume of the liquid. The volume V_1 is called the volume of liquid displaced by the free solid. (Not a strictly accurate term for the case of a body placed in a liquid contained in a vessel of relatively small size, but commonly used and understood.)

It is evident from the way in which P was deduced that it is equal to the total weight of the displaced liquid and that it acts through the c.g. of that displaced liquid. The force P, the upward thrust of all the pressures on the body, is called the force of *buoyancy* of the immersed body and the c.g. of the displaced liquid through which P acts is called the centre of buoyancy. (The British Standard abbreviation is CB, and the diagram letter for this is B. Formerly H was used in books on mechanics.)

Conditions of Equilibrium of a Floating Body. If the body floats either partially or wholly immersed (without any support except the buoyancy) then for equilibrium the buoyancy (P) and the weight (W) of the body must be equal and in the same vertical line so that

$$P = W \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and P is in the vertical line through the c.g. of the whole body. It will be noted that for total immersion the c.g. and centre of buoyancy coincide.

Partial Immersion.—If a homogeneous solid body, of sp. gr. S, floats freely in a liquid of sp. gr. S_1, let V be the volume of the solid and V_1 that of the part immersed. Then $W = wSV$ and the buoyancy $P = wS_1V_1$, namely, the weight of the displaced liquid; and from (3)

$$wSV = wS_1V_1 \text{ and } V_1/V = S/S_1 \quad . \quad . \quad (4)$$

i.e. the ratio of the volume immersed to the whole volume of the solid is S/S_1; and for a flotation in fresh water for which $S_1 = 1$, $V_1/V = S$. Thus a piece of wood of sp. gr. 0·6 will float in fresh water with 0·6 of its volume below and 0·4 above the water surface.

Constraints. If a body does not float *freely* but is partly supported by, say, a suspension or by an upthrust of magnitude T from a support beneath, then

$$P + T = W \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$T = W - P \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or

Thus a body suspended from a spring balance and immersed in a liquid would record an apparent weight not of W but W minus the weight of a volume of liquid equal to that displaced by the solid. In the case of total immersion, the apparent weight (T) would be W minus a weight of liquid of volume equal to that of the suspended solid. Thus for suspension in water and complete immersion,

$$T = W - P = W - W/S = W(1 - 1/S) \quad . \quad . \quad (7)$$

the buoyancy P being equal to the weight of the displaced water, which is 1/S of the weight of the solid body.

This provides a method of finding the sp. gr. of an insoluble solid, for by weighing it in air we find W, and by weighing it suspended in water we find T, and from (7)

$$S = W/(W - T) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

(For high accuracy it is necessary to allow for the buoyancy of air and find the weight in vacuum.)

It is evident that if the body is lighter than water it will need a force T such as the tension of a thread below the body

or a downward thrust to keep it immersed, the buoyancy being greater than the weight. For this constraint

$$T=P-W=W(1/S-1) \quad . \quad . \quad . \quad . \quad (9$$

For immersion in a liquid of sp. gr. S_1, the buoyancy is S_1 times as great as for water; $W/S=P/S_1$ or

$$P=WS_1/S \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and equations (7) becomes

$$T=W(1-S_1/S)=W(S-S_1)/S \quad . \quad . \quad (11)$$

while (9) becomes

$$T=W(S_1/S-1)=W(S_1-S)/S \quad . \quad . \quad (12)$$

It may happen that the centre of buoyancy is not vertically below the c.g. when the body is constrained by an external force T, but the resultant upward force of P and T in equation (6) must be through the c.g. in order to balance W.

Principle of Archimedes. The essential facts about the buoyancy of liquids are embodied in certain historic propositions which amount to the statement that, for a body wholly or partially immersed in a liquid at rest, the resultant upward pressure is equal to the weight of the liquid displaced and acts vertically upward through the c.g. of the displaced liquid. Note that it specifies the magnitude, direction, and position of the resultant pressure.

EXAMPLE. A piece of steel of sp. gr. 7·8 floats in mercury of sp. gr. 13·6. If sufficient water is added just to cover the steel what fraction of the steel is below the surface of the mercury?

Let V be the volume of steel and x be the fraction in the mercury.

Weight of steel = weight of mercury displaced + weight of water displaced

$$Vw \times 7{\cdot}8 = xVw \times 13{\cdot}6 + (1-x)Vw$$

$$12{\cdot}6x = 6{\cdot}8 \qquad x = 6{\cdot}8/12{\cdot}6 = 0{\cdot}54.$$

85. Stability of Floating Body. Metacentre. If a body floats freely, partially immersed, equilibrium for vertical

displacements is stable. A downward displacement increases the buoyancy in excess of the weight, leaving a resultant upward restoring force. And an upward displacement reduces the buoyancy below the weight and leaves a resultant downward restoring force.

We now consider an angular displacement about a horizontal axis. Let Fig. 108 (*a*) represent a cross-section of a floating body such as a boat and let it be symmetrical about an axis perpendicular to the diagram. G is the c.g.

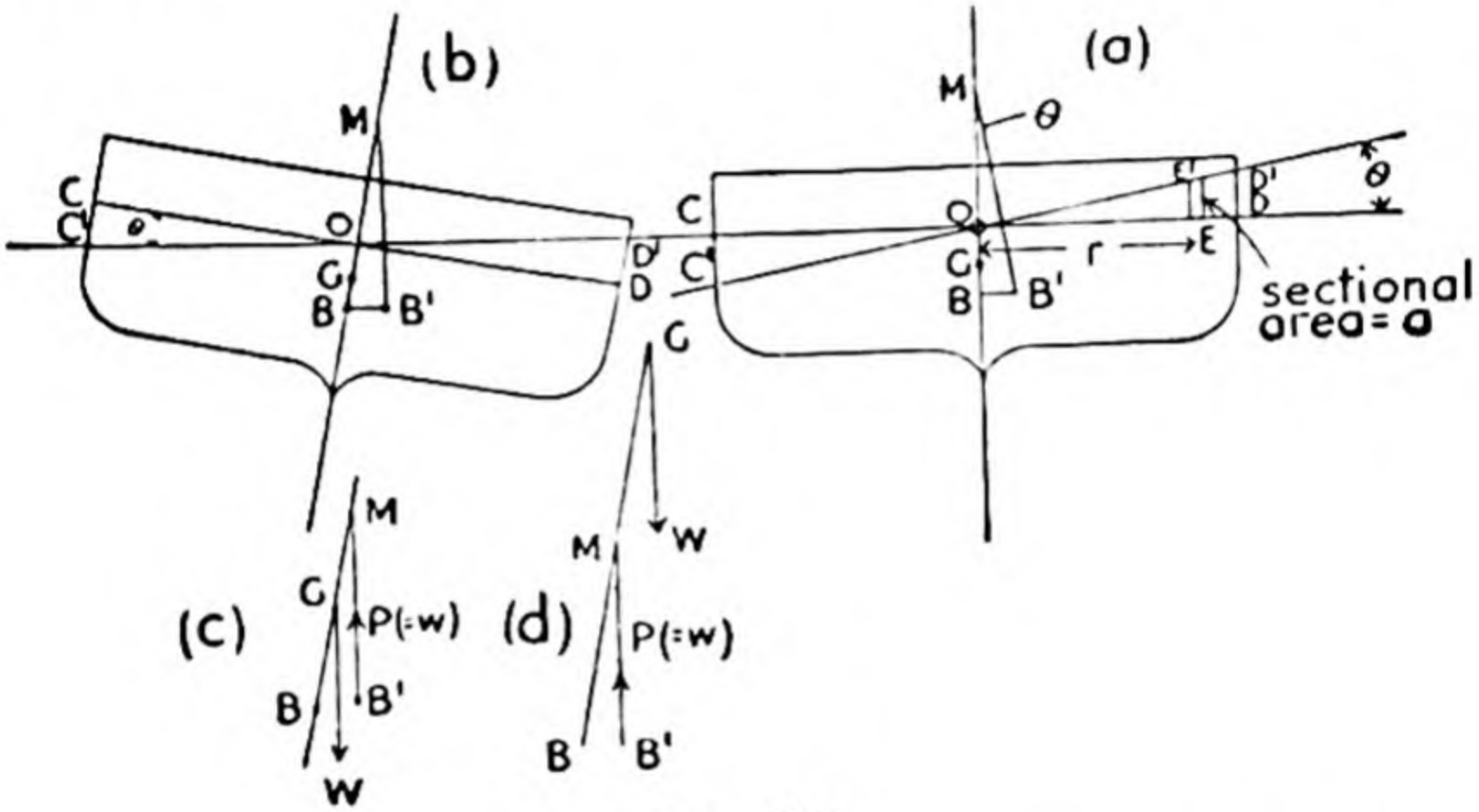

FIG. 108.

of the boat and B the centre of buoyancy, both being in the vertical plane of the diagram. Suppose the body to receive a small clockwise angular displacement θ, as shown at (*b*), Fig. 108, about an axis O such that the total volume immersed is unchanged in magnitude. A vertical line B'M, through B' the new centre of buoyancy, meets the originally vertical line BG produced in M, a point (or rather its limiting position when θ is very small) which is called the metacentre.

If M falls above G as at (*c*), Fig. 108, the weight W of the body acting downward through G, together with the equal force of buoyancy P acting upward through B', will form a couple exerting a counter-clockwise restoring or righting moment which will rotate the body counter-clockwise

about O, and the flotation, initially at least, is stable in respect of angular displacements. But if M were to fall below G, as at (*d*) of Fig. 108, the couple would exert a torque which would rotate the body clockwise, thus increasing the angular disturbance and the equilibrium would be unstable. If M coincides with G, the equilibrium is neutral; there is neither a righting nor an upsetting moment, and the body remains in any new position to which it is (slowly) rotated within the range for which equilibrium may remain neutral.

Metacentric Height. The stability thus depends upon the position of M and we proceed to find this position. At (*a*), Fig. 108, the new position of the water line CD is drawn at C′D′ which shows it relative to the original position of the body. The buoyancy of the displaced liquid changes with the tilt or heel, θ, by the inclusion of the wedge DOD′ and the exclusion of the wedge COC′, thus shifting the centre of buoyancy from B to B′. Consider the change in the moment of the buoyancy, say, about the axis O due to tilting. A prismatic element EE′ of the wedge DOD′ has a cross-sectional area a (an element of the horizontal section of the body through the water surface) and a length θr, if r is the distance of the element from the axis O. Its volume is $\theta r \times a$ and its buoyancy $w\theta ra$, if w is the weight of the liquid per unit volume. (For the volumes of the wedges DOD′ and COC′ to be equal, $\Sigma(ra)$, over the whole area must be zero, *i.e.* the parts of it on opposite sides of the axis must be equal and opposite, hence the axis O is through the centroid of the area $\Sigma(a)$ or A.) The moment of this buoyancy $w\theta ra$ is $w\theta r^2 a$, and for the whole of the wedge, may be written $w\theta\Sigma(ar^2)$. The *change* in moment of buoyancy resulting from the reduction in buoyancy due to lifting the wedge COC′ is equal in magnitude and has the same rotational effect as the depression of DOD′; hence the summation $\Sigma(ar^2)$ is to be taken over the whole area of section A in the water surface and not only for one-half of it. But $\Sigma(ar^2)$ may be written $A \cdot k^2$ (see (5), Art. 9) where k is the radius of gyration of the area A about the axis O.

The *change* in moment of the buoyancy is also represented by $P \times BB'$, where the total buoyancy P is equal to W

when θ is so small that BB′ may be taken as horizontal. (Note that we are considering, not the moments of all the forces exerted on the body, which include its weight, but the change in moment of buoyancy force due to the change in the form of the total displacement V.) Hence,

$$P \times BB' = Ak^2 \times w\theta \quad . \quad . \quad . \quad . \quad (1)$$

And since $P = Vw$ and $BB' = \theta \times BM$, when θ is small,

$$Vw \times BM \times \theta = Ak^2 \times w\theta$$

$$BM = Ak^2/V \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This determines the position of the point M. The length GM is generally called the metacentric height. For stability, BM must be greater than BG, and for neutral equilibrium (in respect of a tilt), BM is equal to BG. The higher M is above G, the greater the stability.

The metacentric height GM is found experimentally by moving a weight w, say, across the axis of tilt for a distance, say, d. This gives a tilting moment of wd. The movement of the c.g. is then wd/W

and the angle of heel $\theta = wd/\{W \times (GM)\}$ radians,

a quantity practically measured by the shift s at the end of plumbline of length l;

hence $\theta = s/l = wd/\{W(GM)\}$ or $GM = \frac{wd}{W} \cdot \frac{l}{s}$

EXAMPLE 1. How long may the axis of a cylinder of radius a be if it is to float freely in stable equilibrium with the axis vertical.

Let S be the sp. gr. of the material and h be the required length of axis. Then the length of axis submerged from (4), Art. 84, is Sh. The depth of the centre of buoyancy below the surface is $\frac{1}{2}S \cdot h$. The height of c.g. above the centre of buoyancy $= \frac{1}{2}h - \frac{1}{2}Sh = \frac{1}{2}h(1 - S)$. Then the volume immersed being $S \times h \times \pi a^2$, height of metacentre above centre of buoyancy

$$= BM = Ak^2/V = \pi a^2 \times \tfrac{1}{4}a^2/(\pi a^2 \times Sh) = \tfrac{1}{4}a^2/Sh$$

and for stability, the c.g. (G) must not be above the metacentre M, or

$\frac{1}{2}h$ (1—S) must not exceed $\frac{1}{4}a^2/Sh$

or h^2 „ „ $\frac{1}{2}a^2/S(1—S)$

h „ „ $a/\sqrt{2S(1—S)}$

e.g. if $S=\frac{1}{2}$, h must not exceed $a\sqrt{2}$

EXAMPLE 2. A pontoon is 18 ft. long by 9 ft. wide and the total weight is 27 tons. Find the position of the metacentre for rolling in seawater. How high may the c.g. be so that the pontoon shall not overturn. Take seawater as occupying 35 cu. ft. per ton.

Vol. of displacement $=27\times35=945$ cu. ft.

Depth of displacement $=945$ cu. ft./(18×9) sq. ft. $=5{\cdot}8333$ ft.

k is least for the central axis parallel to the long sides and for this,

$$Ak^2=18\times9^3/12$$

$$BM=Ak^2/V=18\times729/(12\times945)=1{\cdot}1571 \text{ ft.}$$

Height of M above base $=1{\cdot}1571+\frac{1}{2}\times5{\cdot}8333=4{\cdot}074$ ft.

This gives the limit of the height of c.g. above base for stability.

EXAMPLES VIII

1. Find the total pressure on a submerged vertical sluice gate 4 feet by 3 feet, the longer sides being horizontal, when the top side is 2 feet below the surface of fresh water.

2. A circular door 4 feet diameter has its centre 3 feet below the surface of fresh water. Find the total pressure on the door and the depth of the centre of this pressure.

3. A piece of wood weighing 20 lb. floats freely in water with 60 per cent. of its volume below the surface. It takes 15·3 lb. of a metal attached to the wood just to submerge it. Find the sp. gr. of the wood and of the metal.

4. A ship weighs 2000 tons and has vertical sides at the water line where the area of horizontal section is 20,000 sq. ft. How much will she rise on passing from fresh to sea water if this weighs 64 lb. per cu. ft., and fresh water 62·4 lb. per cu. ft. ?

5. A uniform rod 3 feet long is freely hinged at its lower end 2 ft. below the surface of water. If the rod rests in a position inclined 30° to the vertical, find the sp. gr. of the material of the rod.

6. A vessel of 1000 tons displacement has its transverse metacentre 6 feet above the centre of buoyancy and the c.g. 3·5 feet above the centre of buoyancy. If a load of 10 tons is moved transversely 12 feet across the deck, find the consequent angle of heel of the vessel.

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CHAPTER IX

HYDRAULICS

86. Transmission of Pressure and Energy. We now return to the equality of pressure in all parts of a body of fluid except in so far as there is some variation with depth, due to the weight of the fluid, particularly with a heavy liquid. But in hydraulic machinery the pressure to which the water or other fluid (such as oil) is subjected is often so high that we can neglect any differences due to slight differences in depth at different parts of any containing vessel, and we shall do so in what follows.

The equality of intensity of pressure makes water or other liquids under pressure a very convenient medium

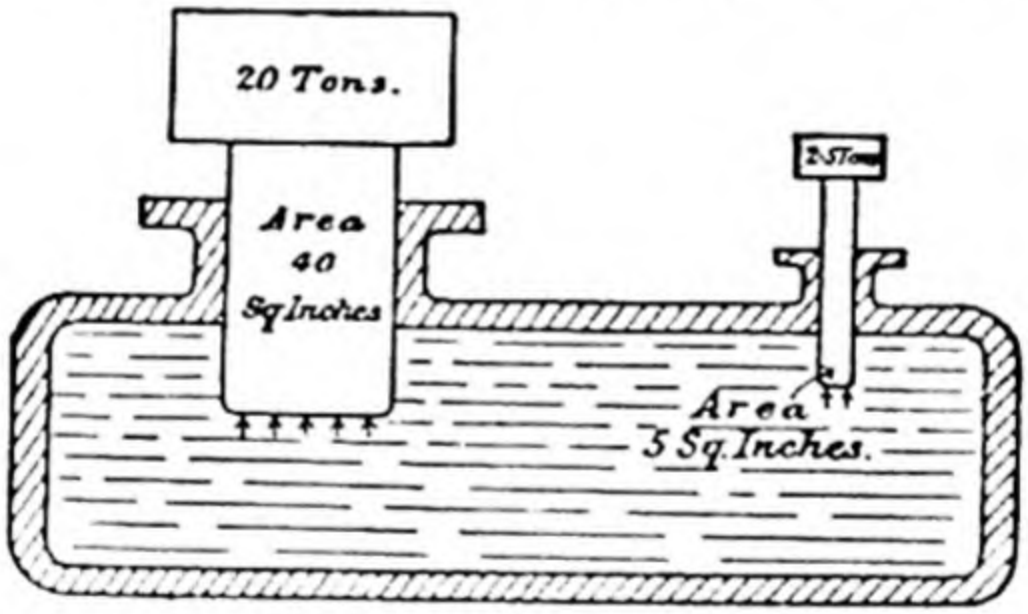


FIG. 109.—Transmission of pressure by liquid.

for transferring the force exerted in one place to another place where the use of force is required. And it also allows a small force exerted through a large distance to overcome a large resistance through a smaller distance as an alternative to using some form of gearing such as is used in the various lifting machines.

The principle will be most easily understood from an ideal case diagrammatically shown in Fig. 109. This represents two frictionless water-tight pistons or plungers subject to the pressure of the water contained in a vessel capable of holding it under considerable pressure. If the

smaller plunger has an area of cross-section of 5 sq. in. and carries a load of 2·5 tons, the total force divided by this area, or 2·5 tons/5 sq. in., gives a pressure of 0·5 ton per square inch which the plunger exerts downward on the water, and as a reaction, the water exerts a total force of 2·5 tons at 0·5 ton per square inch upward on the plunger. This pressure of 0·5 ton per square inch will also be exerted on the larger plunger, and if its area of cross-section is, say, 40 sq. in., the total load it can lift (or other resistance it can overcome) is 0·5 ton per square inch $\times$40 sq. in. $=$20 tons.

If friction be neglected then, with such an arrangement, a force of 2·5 tons can lift a load of 20 tons, that is, 8 times as much as the "effort" of 2·5 tons, because the larger plunger has an area of cross-section 40/5 or 8 times as large as the small one. Since the volume of water displaced by the smaller plunger must be equal to that previously occupied by the larger one before it was raised (for the total volume of water does not change in the operation), the vertical movement of the small plunger must be 8 times as great as that of the large one. For example, if the small plunger comes down 10 in., it displaces 10 in. $\times$5 sq. in. $=$50 cu. in. of water, and to accommodate this water the larger plunger must vacate 50 cu. in. of space. And since its area of cross-section is 40 sq. in., it must move up by 50 cu. in./40 sq. in. $=$1·25 in., which is one-eighth of the motion of the small plunger. Thus, regarded as a lifting apparatus, the velocity ratio or movement ratio is 8 and the mechanical advantage 20 tons/2·5 tons is also 8, friction being assumed to be zero.

Put in symbols, let W_1 be the effort in pounds, A_1 sq. ft. the area of cross-section of the effort plunger, W_2 the load in pounds, and A_2 sq. ft. the area of cross-section of the load plunger. Then if the load moves up l_2 ft. and the effort moves downwards l_1 ft., since the volumes must be equal,

$$(A_2 \times l_2) \text{ cu. ft.} = A_1 \times l_1 \text{ cu. ft.} \quad . \quad . \quad (1)$$

or

$$\frac{l_1}{l_2} = \frac{A_2}{A_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is the velocity ratio.

Fig. 109 illustrates the principle of the hydraulic press and the hydraulic lifting jack, in both of which the effort

is often applied by a lever to a small plunger or ram, and acting through a large distance, can lift a large load.

Hydraulic power is transmitted by forcing water at high pressures along pipes from a central pumping station. It is particularly useful where the demand for power is intermittent (there being no loss of heat energy when the machinery is not working), and where great forces have to be exerted slowly, and through short distances.

Hydraulic Press. The construction of a hydraulic press

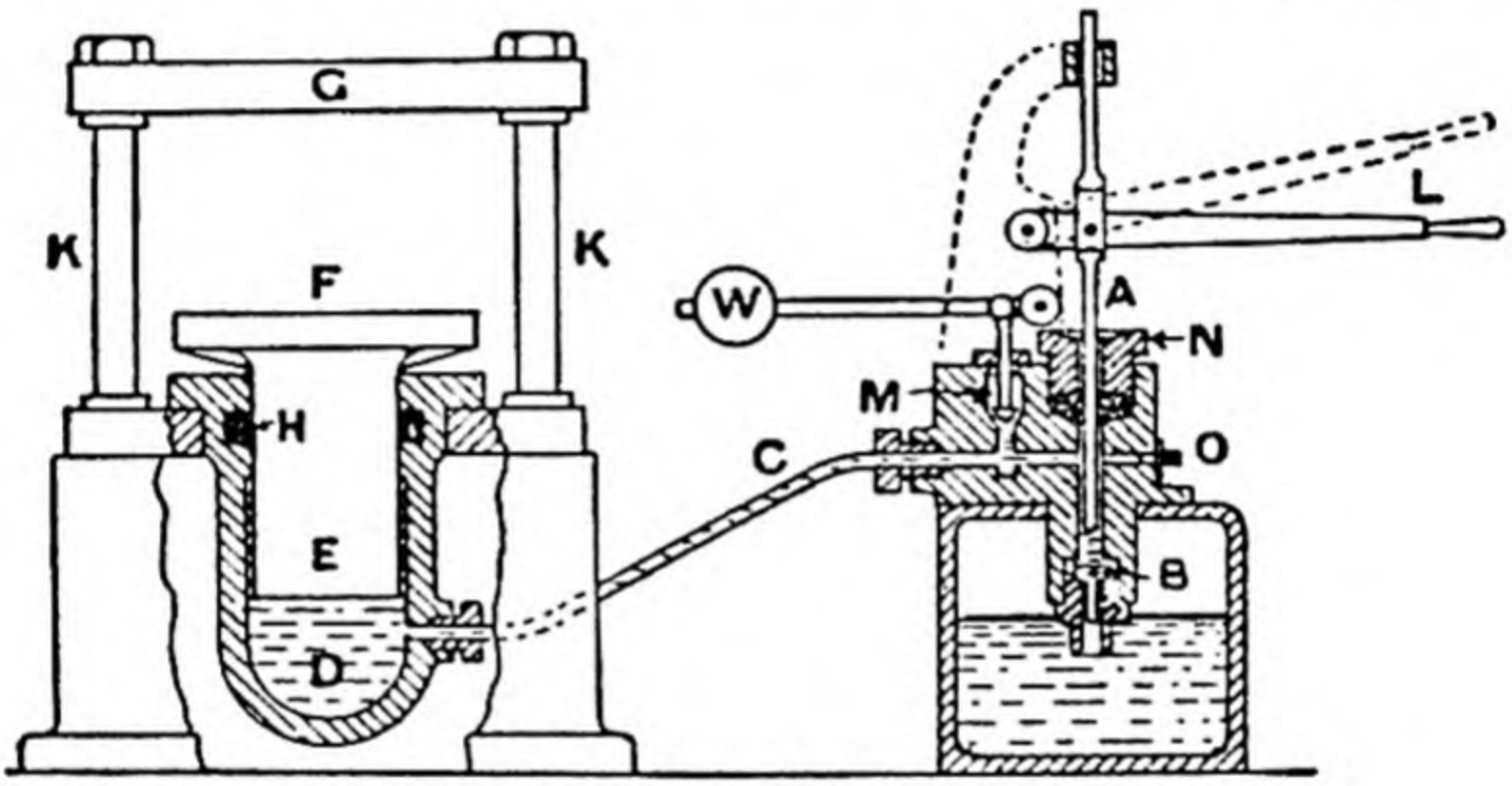


FIG. 110.—Hydraulic press.

is shown in Fig. 110, and the method of working is as follows:

On the upward stroke of the lever L, the plunger A is raised. The valve B is raised, and water flows past it to follow the plunger. On the downward stroke of L, B closes, and water is forced out along the pipe C into the ram cylinder D, pushing up the ram E. The top of the ram forms a platform, F, on which the material to be pressed is placed. Columns K carry another fixed platform G, and as F is forced upwards, the material is compressed between F and G. To prevent water leaking out of the ram cylinder D, the leather ring H is inserted in an annular recess, as shown. The greater the pressure of the water (and therefore the upward force on F), the tighter is the joint formed by H. The cup leather H is placed in a recess, which is turned in the ram cylinder D in such a way that the water under pressure can pass into the annular

space inside the leather. The result is, that the greater the water pressure, the tighter the leather is pressed against the ram E, and the better the joint.

As the material is compressed more and more between F and G, the pressure of the water rises, and to prevent damage to the machine, a safety-valve, M, is fitted. This valve opens when the pressure rises to a dangerous height, and allows water to escape, and so relieves the pressure. When the material is compressed as much as is desired, the ram E is allowed to return again by its own weight, by opening the valve at O, and so allowing water to escape.

The base of the ram cylinder D is made hemispherical because that shape is the strongest to resist great internal pressures.

Transmission of Energy. The load of 2·5 tons in Fig. 109 loses potential energy by coming to a lower level, and the load of 20 tons gains an equal amount of potential energy by being lifted, and the work done by the effort and that done on the load is a measure of the transfer of energy from one to the other. And quantitatively this amount of energy can be stated in terms of the number of units of work done by one plunger and on the other. Putting this in symbols and using those already employed above, let p be the intensity of pressure in pounds per square *foot*.

$$\text{Then } p = \frac{\text{total force}}{\text{area}} = \frac{W_2 \text{ lb.}}{A_2 \text{ sq. ft.}} = \frac{W_1 \text{ lb.}}{A_1 \text{ sq. ft.}} . \quad (3)$$

For axial movements l_2 and l_1 feet, work done on the water

$$= W_1 l_1 = p \text{ lb./sq. ft.} \times A_1 \text{ sq. ft.} \times l_1 \text{ ft.} = p A_1 l_1 \text{ ft.-lb.} \quad (4)$$

and this is equal to work done by the water

$$= W_2 l_2 = p \text{ lb./sq. ft.} \times A_2 \text{ sq. ft.} \times l_2 \text{ ft.} = p A_2 l_2 \text{ ft.-lb.} \quad (5)$$

In these two equal expressions for the work done, the parts $A_1 l_1$ and $A_2 l_2$ both represent the volume in cubic feet of water displaced. Let this be represented by V (cu. ft.), then the

$$\text{work done} = pV \text{ ft.-lb.} \quad . \quad . \quad . \quad . \quad (6)$$

$$= \text{pressure per } \textit{square foot} \times \text{volume in } \textit{cubic feet}. \quad (7)$$

This expression for work done holds good for any fluid entering at constant pressure p a space of volume V, whatever the shape of that space. It cannot enter a space at constant pressure unless it is overcoming a resistance driving something in front of it and doing work in making room for itself, and the product pV is the measure of that work done. Note particularly that the product of p in pounds per square *foot* and V in cubic *feet* gives work in foot-pounds. Similarly, p in pounds per square *inch* and V in cubic *inches* would give work in inch-pounds. There is great liability to numerical error here unless consistent units are used.

The expression pV for work done by a fluid at constant pressure p applies equally to a liquid and a gas.

It has another form which is often useful for 1 lb. of fluid. If the fluid weighs w lb. per cu. ft., the volume of 1 lb. is $1/w$ cu. ft. Hence the work done by 1 lb. of a fluid against constant resistance at a pressure p lb. per sq. ft. is given by

$$\text{Work per lb.} = p \times \text{volume per lb.} = p/w \text{ ft.-lb.} \quad . \quad (8)$$

and this represents the energy transmitted per pound of water moved, a form which we shall use later (Art. 90).

Note that for the transmission of the energy, movement is essential. The water in fact transmits the energy in much the same way as a solid rod, such as a piston rod, at a rate of p/w ft.-lb. per pound of water which crosses any transverse section along the path of transmission, *i.e.* between the two plungers in Fig. 109.

EXAMPLE 1. A hydraulic press is operated by an effort of 40 lb. exerted at right angles to a lever and 30 in. from its fulcrum. The plunger is pushed also at right angles to the lever and 2 in. from the fulcrum; it is 1 in. in diameter. The ram or large plunger is 10 in. in diameter. Find the force which can be exerted by the press if the efficiency of the apparatus at this load is 85 per cent., *i.e.* if 15 per cent. of the energy is wasted in friction.

Firstly neglecting frictional losses, from the principle of moments,

$$40 \text{ lb.} \times 30 \text{ in.} = \text{force on plunger} \times 2 \text{ in.}$$

$$\text{Force exerted on plunger by lever} = 40 \text{ lb.} \times \frac{30 \text{ in.}}{2 \text{ in.}}$$
$$= 600 \text{ lb.}$$

$$\text{Area of plunger} = 0{\cdot}7854 \times 1 \text{ sq. in.} = 0{\cdot}7854 \text{ sq. in.}$$

$$\text{Pressure intensity in water} = \frac{600 \text{ lb.}}{0{\cdot}7854 \text{ sq. in.}}$$
$$= 764 \text{ lb./sq. in.}$$

$$\text{Area of ram} = 0{\cdot}7854 \times 10^2 = 78{\cdot}54 \text{ sq. in.}$$

Total pressure on ram

$$= 764 \text{ lb./sq. in.} \times 78{\cdot}54 \text{ sq. in.} = 60{,}000 \text{ lb.}$$

Allowing for 15 per cent. loss,

$$\text{Total pressure} = 60{,}000 \text{ lb.} \times 0{\cdot}85 = 51{,}000 \text{ lb.}$$

EXAMPLE 2. In a hydraulic jack the leverage of the handle is 15 to 1. Calculate the ratio of the diameter of the large plunger or ram to that of the small plunger worked by the handle if a force of 20 lb. on the handle is to lift a load of 2000 lb., assuming that the efficiency is 70 per cent.

$$\text{Force on plunger} = 15 \times 20 \text{ lb.} = 300 \text{ lb.}$$

$$\text{Pressure in water } p = \frac{300 \text{ lb.}}{0{\cdot}7854 d^2} \text{ lb. per sq. in.}$$

where d = diam. of plunger in inches.

The effective force for lifting is 70 per cent. of $p \times 0{\cdot}7854 \mathrm{D}^2$, where D = diam. of ram in inches, and this must be equal to 2000 lb. Hence,

$$0{\cdot}7 \times \frac{300 \text{ lb.}}{0{\cdot}7854 d^2} \times \frac{0{\cdot}7854 \mathrm{D}^2}{1} = 2000 \text{ lb.}$$

$$\frac{\mathrm{D}^2}{d^2} = \frac{2000}{0{\cdot}7 \times 300} = \frac{2{\cdot}1}{20} = 9{\cdot}52$$

$$\frac{\mathrm{D}}{d} = \sqrt{9{\cdot}52} = 3{\cdot}1$$

EXAMPLE 3. Hydraulic power is supplied from pipes in which the water is at 500 lb. per sq. in. pressure. How

many gallons per hour will be required to operate a motor or machine working at 20 h.p. if the efficiency is 70 per cent. ?

The work to be done in 1 hr.

$$=20\times33{,}000 \text{ ft.-lb.}\times60=39{,}600{,}000 \text{ ft.-lb.}$$

Work obtainable from V cu. ft.

$$=\frac{70}{100}\times\frac{500 \text{ lb.}}{\text{sq. in.}}\times\frac{144 \text{ sq. in.}}{\text{sq. ft.}}\times V \text{ cu. ft.}$$

$$=\frac{70}{100}\times500\times144V \text{ ft.-lb.}$$

and this must give 39,600,000 ft.-lb., hence

$$V=\frac{39{,}600{,}000\times100}{70\times500\times144}=786 \text{ cu. ft.}$$

and at 6·24 gallons to 1 cu. ft., the quantity required is

$$786 \text{ cu. ft.}\times6{\cdot}24 \text{ gal. per cu. ft.}=4734 \text{ gallons.}$$

87. Suction or Lift Pump. This common piece of hydraulic apparatus illustrates some of the principles of hydrostatics. Consider first what suction means. If a tube is placed with one end in a bowl of water and air is drawn out of the top end of the tube, the pressure in the tube falls below that of the atmosphere: this is called forming a "partial vacuum." But the water in the bowl has exerted upon it the pressure of the atmosphere which rests on its surface. This pressure, being now greater than that in the upper part of the tube, drives the water up the tube until the weight of water standing in the tube just balances the difference between the upward pressure (atmospheric) of the water at the bottom of the tube and the reduced air pressure in the upper part of the tube. All "suction" is of this nature, that is, the pressing of a liquid forward by a full atmospheric pressure acting against a reduced pressure; there is no appreciable pulling of a liquid. In other words, there is practically no tension possible in a fluid.

A suction pump is shown diagrammatically in Fig. 111. It consists of a cylindrical pump barrel B, in which an air-tight plunger or bucket A is driven up and down by

hand or other agency. This bucket has one or more valves C which can only open upwards to allow the water to pass through them from the lower to the upper side. A pipe F attached to the bottom of the pump barrel B leads down to the water in the well, the lower end of F being always submerged. At the top of the pipe F is a valve E

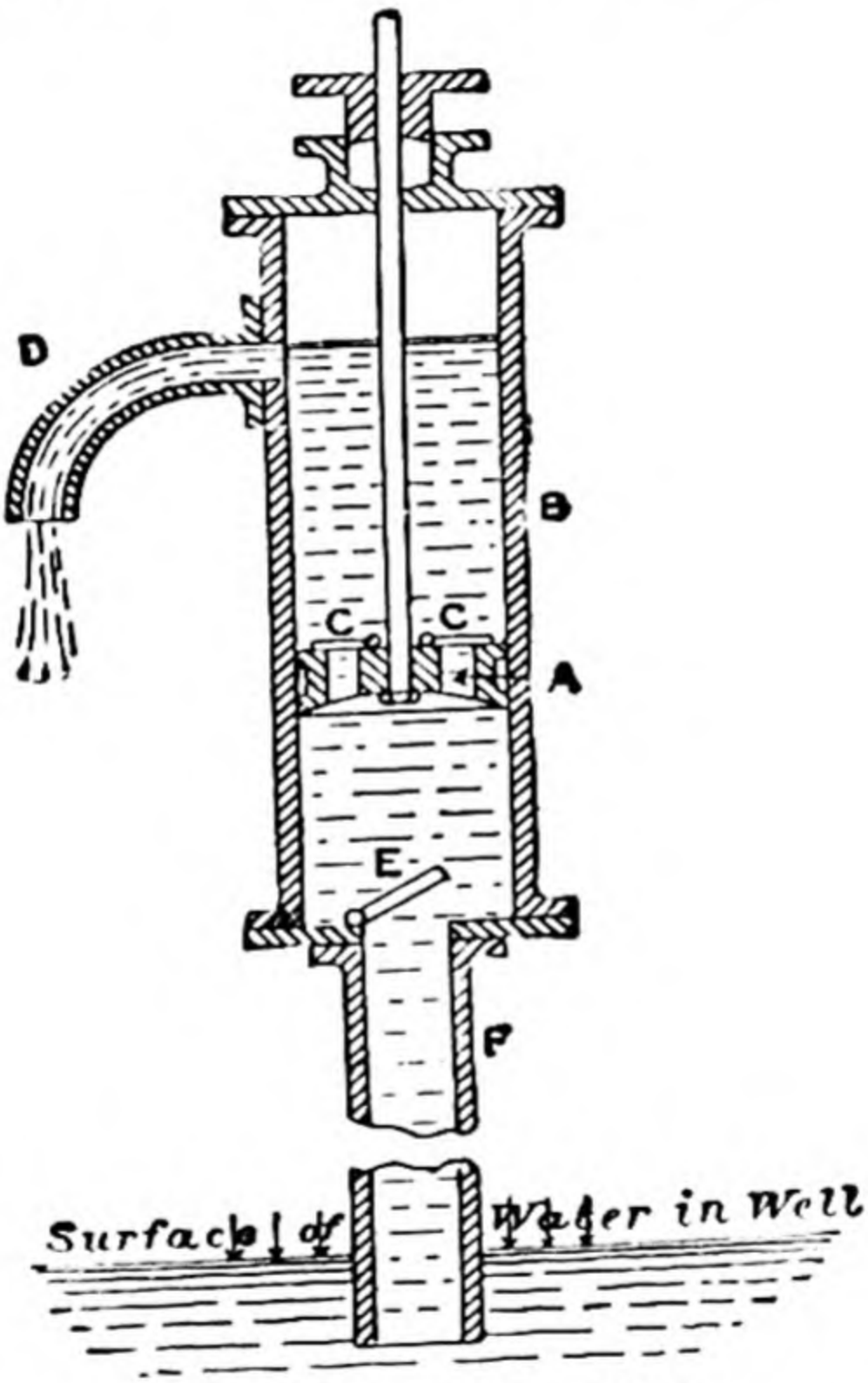


FIG. 111.—Suction pump.

which opens upwards to allow ascending water to pass into the pump barrel B.

When the pump is started, B and F are full of air, and water will not be delivered till this is cleared away. Suppose the bucket A is at the top of its stroke. As it is moved downwards, E is closed, and the valves C open and

so allow the air in the lower part of the barrel to pass through the bucket. On the upward stroke of the bucket, the valves C close and above E the pressure is lowered below that of the atmosphere; the valve E opens and air is drawn out of F with the result that the pressure in F also is lowered below that of the atmosphere. But the water in the well has the pressure of the atmosphere pressing down on its surface and transmitted normally in all directions in the water. Consequently some water is driven up the pipe F by this pressure. The amount will be such that the water, together with the (reduced) air pressure above it, exerts a downward force equal to the upward force exerted by the water in the well due to the pressure of the atmosphere. This process is repeated with successive downward and upward strokes till the space below the bucket is full of water instead of air, and then water passes through the valves E and C on the upward and downward strokes respectively. After this, water which has passed through the valves C on a downward stroke is lifted directly by the bucket on its upward stroke, this final lift not being by suction (or atmospheric pressure) but by direct lift. The water, after this direct lift, flows out of a spout D.

Since the water is driven up the pipe by the pressure of the atmosphere, or the excess of it over that in the pump barrel, the height to which water can be thus lifted is limited by this available pressure.

We have seen in the example in Art. 80 that the limit of height for 14·7 lb. per sq. in. pressure of the atmosphere is 34 ft. This height of lift could only be achieved if we could reduce the pressure at the top of the lift pipe to zero. For various reasons, particularly because we cannot make the joints, valves, etc., perfectly air-tight, we can only get a moderately low pressure by such an apparatus and the direct suction lift in practice seldom exceeds 26 ft. There is no corresponding limit to the remaining lift, that is, to the distance from the bottom of the pump barrel at E to the spout at D. But for practical and mechanical reasons this lift is usually short.

Volume Delivered. If the area of cross-section of the barrel and plunger or bucket is A sq. in., say, and the stroke is l in., the volume swept through at each stroke

is $A \times l$ cu. in., and this would be the nominal delivery on each *upward* stroke. But actually, apart from leakages after valves are closed, some water slips back through the valves while they are closing, for valves cannot act instantaneously, particularly when, as shown, they act automatically with changes of pressure. The actual delivery is therefore less than the nominal by the amount of this "slip."

EXAMPLE 1. The barrel of a suction pump is 5 in. diameter and the stroke is 8 in. How many upward strokes of the bucket will be required to lift 1000 galls. of water if there is 12 per cent. of slip?

Volume lifted per stroke

$$= \frac{88}{100} \text{ of area of section} \times \text{length of stroke}$$

$$= \frac{88}{100} \times \frac{0{\cdot}7854 \times 5^2}{144} \text{ sq. ft.} \times \frac{8}{12} \text{ ft.}$$

$$= 0{\cdot}0714 \text{ cu. ft.}$$

Volume to be lifted

$$= \frac{1000 \text{ gal.}}{6{\cdot}24 \text{ gal./cu. ft.}} = 160{\cdot}3 \text{ cu. ft.}$$

Number of strokes required

$$= \frac{160{\cdot}3}{0{\cdot}0714} = 2245.$$

EXAMPLE 2. If the pump in Example 1 delivers water 80 ft. above the bucket, find the total pull in the rod which lifts the bucket, in addition to that from its own weight. (Neglect any acceleration of the water and any partial vacuum under the bucket.)

Volume of water above the bucket

$$= \text{area} \times \text{length of column}$$

$$= \frac{0{\cdot}7854 \times 25}{144} \text{ sq. ft.} \times 80 \text{ ft.}$$

$$= 10{\cdot}91 \text{ cu. ft.}$$

Tension in rod =weight of water
=10·91 cu. ft. ×62·4 lb./cu. ft.
=681 lb.

88. Force Pump. This differs from the lift pump in having no valves in the plunger. It is shown diagrammatically in Fig. 112. The plunger A is solid and on its upward stroke it draws water into the barrel B through the suction valve C, just as in the previous case, *i.e.* by the pressure of the atmosphere. But usually this suction lift is short, not approaching the full lift of 34 ft. corresponding to the full atmospheric pressure. During this upward stroke the delivery valve D is kept closed by the pressure of the column of water in the delivery pipe E. On the return or downward stroke of the plunger, the pressure of the water above it closes the suction valve C and the pressure in the barrel becomes greater than that of the water in E and opens the delivery valve D, through which the water is forced up the delivery pipe E. As in the previous type there is, of course, some "slip" of water at the valves. The greater part of the lift is usually effected by the downward thrust of the plunger. Thus the lift is not limited by the magnitude of the atmospheric pressure.

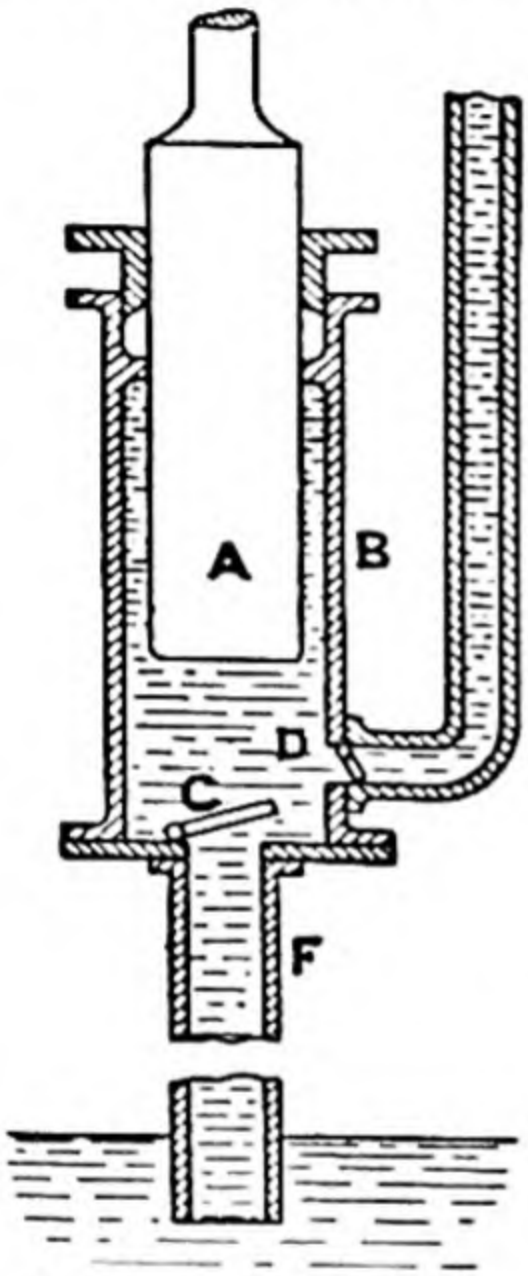


FIG. 112.—Force pump.

EXAMPLE 1. A force pump delivers water against a head of 180 ft. What average force will be required on the pump plunger during the delivery stroke if the mechanical efficiency is 85 per cent., *i.e.* if 15 per cent. of the energy supplied is wasted in friction?

Pressure due to 180 ft. head of water
=180 ft. ×62·4 lb./cu. ft.
=11,230 lb. per sq. ft.

$$\text{Area of plunger} = \frac{0{\cdot}7854 \times 5^2}{144} \text{ sq. ft.}$$

$$= 0{\cdot}1363 \text{ sq. ft.}$$

Force exerted by plunger on water

$$= 0{\cdot}1363 \text{ sq. ft.} \times 11{,}230 \text{ lb./sq. ft.}$$

$$= 1530 \text{ lb.}$$

Driving force exerted on plunger

$$= \tfrac{100}{85} \times 1530 = 1800 \text{ lb.}$$

EXAMPLE 2. Find the horse-power required for the pump in Example 1, neglecting the work done in the suction stroke if the pump has a stroke of 8 in. and makes 45 working strokes per minute.

Work per stroke = force exerted on plunger × stroke

$$= 1800 \text{ lb.} \times \tfrac{8}{12} \text{ ft.} = 1200 \text{ ft.-lb.}$$

Work done per minute

$$= 1200 \text{ ft.-lb.} \times 45 = 54{,}000 \text{ ft.-lb.}$$

$$\text{power required} = \frac{54{,}000 \text{ ft.-lb./min.}}{33{,}000 \text{ ft.-lb./min.}}$$

$$= 1{\cdot}64 \text{ h.p.}$$

EXAMPLE 3. If the slip in the pump in Examples 1 and 2 is 10 per cent., how many gallons of water will be delivered per minute?

Volume displaced by plunger

$$= \text{area} \times \text{stroke}$$

$$= 0{\cdot}1363 \text{ sq. ft.} \times \tfrac{8}{12} \text{ ft.}$$

$$= 0{\cdot}0909 \text{ cu. ft.}$$

$$\text{Water delivered per stroke} = \tfrac{90}{100} \times 0{\cdot}0909 \text{ cu. ft.}$$

$$= 0{\cdot}0818 \text{ cu. ft.}$$

$$\text{Water delivered per minute} = 0{\cdot}0818 \times 45 = 3{\cdot}68 \text{ cu. ft.}$$

$$= 3{\cdot}68 \text{ cu. ft.} \times 6{\cdot}24 \text{ gal./cu.ft.}$$

$$= 23 \text{ gallons.}$$

89. Water in Motion. So far we have considered liquids in slow motion when transmitting energy as well as pressure, and forces necessary to accelerate the liquid have been neglected. But when in motion, because it has mass, a liquid carries in itself kinetic energy which, though negligible at very low speeds, becomes important when the motion is rapid, since it is proportional to the square of the speed. A few applications of mechanics of motion to liquids are important and form excellent illustrations of

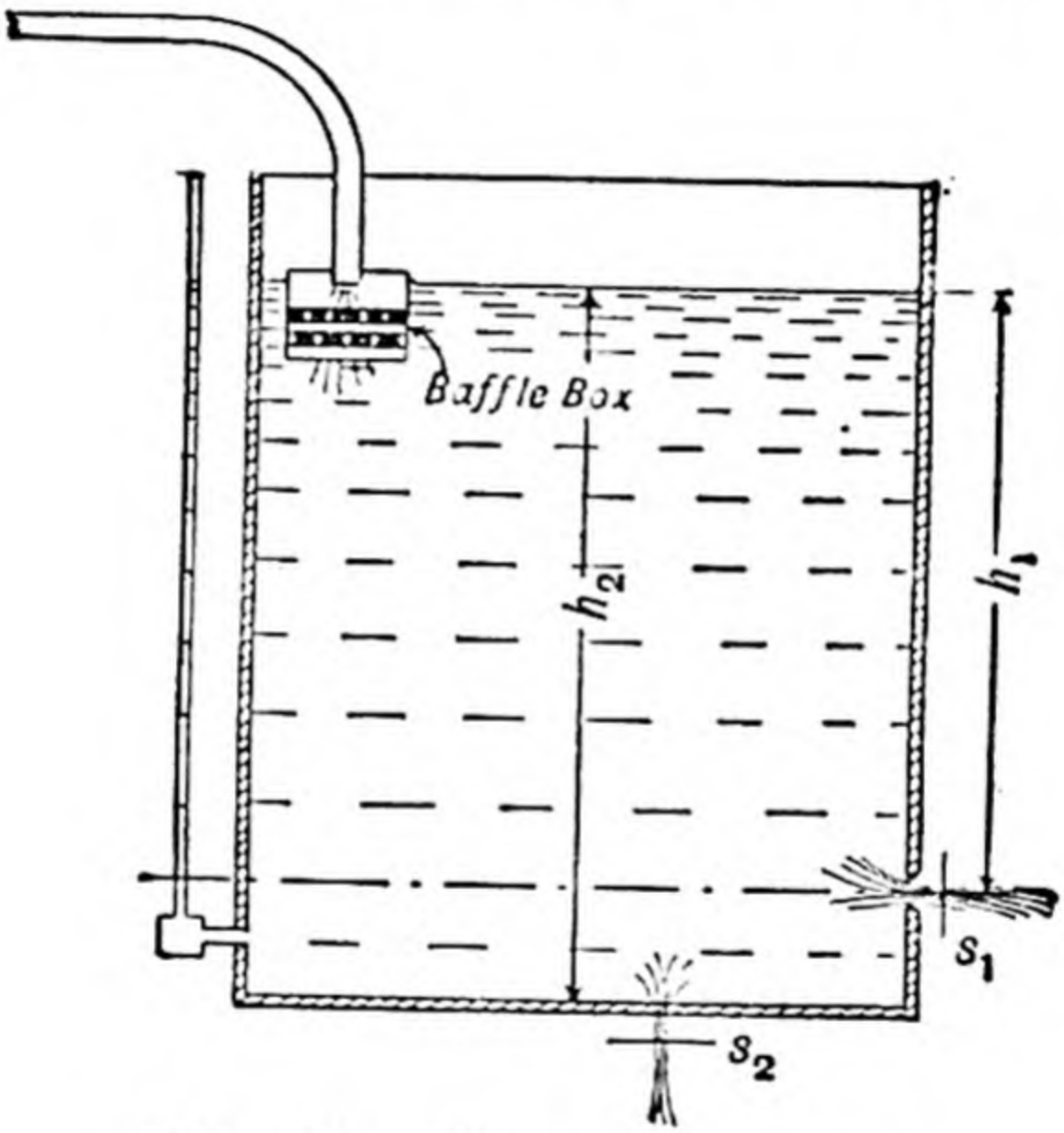


FIG. 113.—Flow from orifices.

the principles of mechanics. And they may be treated very simply if we limit ourselves to cases where friction and viscosity are neglected. In many problems relating to the flow of water, friction is not negligible, and we refer to friction in pipes later (Art. 91).

Flow from an Orifice. We first consider the flow of water through a hole or orifice in a vessel containing water under constant pressure. Fig. 113 represents an experimental tank in which the outflow is constantly replenished by an

inflow so arranged as to give a large volume of nearly still water maintained at a constant level. Experimental orifices of various shapes may be used, but a circular one is commonest; they are usually sharp edged, as shown. This reduces the friction of flow and gives a clear stream with a fairly definite boundary instead of a broken stream of which it would be impossible to measure the diameter. The stream, on issuing from the orifice, contracts to a smaller cross-section than that of the orifice and the ratio which the minimum sectional area at S_1 or S_2 bears to the full area of the orifice is called the coefficient of contraction of the orifice. (It can be measured with some difficulty.) If the level of the still surface in the tank is h_1 or h_2 ft. above the centre of an orifice (Fig. 113), the velocity of outflow is nearly equal to that of a body falling freely through a height h_1 or h_2 ft., viz. $\sqrt{2gh_1}$ or $\sqrt{2gh_2}$ ft./sec. (see (3), Art. 64). The ratio which the actual velocity v at S_1, say, bears to the velocity $\sqrt{2gh_1}$ ft./sec. due to falling freely h_1 ft. is called the coefficient of velocity, which is generally about 0·97. The volume V in cubic feet flowing per second would be that of a cylinder having a cross-sectional area a_1 sq. ft. equal to that of the stream at S_1 and a length v ft. or

$$V = a_1 \times v \text{ or about } 0{\cdot}97 a_1 \sqrt{2gh_1} \text{ cu. ft./sec.}$$

And if a = area of orifice in square feet, the coefficient of contraction is a_1/a, so that the volume flowing may be written

$$V = a \times a_1/a \times v$$
$$= a \times \text{coefficient of contraction} \times \text{coefficient of velocity} \times \sqrt{2gh_1} \quad . \quad (1)$$

and the product of these two coefficients is called the coefficient of discharge, so that

$$V = a \times \sqrt{2gh_1} \times \text{coefficient of discharge}$$

or

$$V = ka\sqrt{2gh_1} \text{ cu. ft./sec.} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where k is the coefficient of discharge. This coefficient of discharge k, which is about 0·62 for a circular sharp-edged orifice, is determined by measuring the volume V and the

linear dimensions a and h_1. With a coefficient of contraction of, say, 0·64 we have

coefficient of discharge
=coefficient of contraction × coefficient of velocity . (3)

e.g. $0{\cdot}62 = 0{\cdot}64 \times 0{\cdot}97$.

The coefficient of discharge is easily determined from V, and the coefficient of contraction can be measured by special apparatus. The coefficient of velocity can be found by calculation from the above relation.

The fact that with negligible friction the velocity of outflow would be equal to that of falling freely through a height h_1 ft. could be deduced from the principle of the conservation of energy. For since the incoming water is at a height h_1 above the orifice, it has a potential energy of h_1 ft.-lb. per lb. above that at outflow. And if there is no loss this must be the kinetic energy per pound at outflow, which is $v^2/2g$ (see (5), Art. 64, when W=1). Hence

$$v^2/2g = h_1$$

or
$$v = \sqrt{2gh_1} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Velocity from Pressure. But we may look at the matter in another way. At a depth h_1, the pressure on a square foot is wh_1 lb., where w is the weight of a cubic foot of the liquid, or

$$p = wh_1 \text{ (as in Art. 79)}$$

and this pressure does work on the outflowing stream, which is $p \times$ volume. And if we consider 1 lb. of water, its volume is $1/w$ cu. ft., so that the work done on it is p/w ft.-lb., which must be equal to the kinetic energy or

$$\frac{p}{w} = \frac{v^2}{2g} \quad \text{or} \quad v = \sqrt{\frac{2gp}{w}} \text{ ft./sec. . . . (5)}$$

and since $p = wh_1$,

$$h_1 = v^2/2g \quad \text{or} \quad v = \sqrt{2gh_1} \text{ ft./sec. . . (6)}$$

The relation (5) would give the velocity of flow from an orifice in a vessel containing water at pressure p lb. per

sq. ft., whether due to a depth of water h_1 ft., or to the force on a pump plunger, or any other cause.

EXAMPLE 1. Find the diameter of a sharp-edged circular orifice to discharge 2400 gal. per hour under a head of 40 ft. of water.

$$\text{Discharge} = 2400 \text{ gal./hr.} \times (1/6{\cdot}24) \text{ cu. ft./gal.}$$
$$= 385 \text{ cu. ft./hr.}$$
$$= 0{\cdot}107 \text{ cu. ft./sec.}$$

If a is the area of orifice in square feet, then, assuming a coefficient of discharge of 0·62, the discharge in cubic feet per second under a head of 40 ft. is

$$0{\cdot}62 \times a \times \sqrt{2 \times 32{\cdot}2 \times 40} = 0{\cdot}107 \text{ cu. ft./sec.}$$
$$0{\cdot}62 \times a \times 50{\cdot}8 \text{ ft./sec.} = 0{\cdot}107 \text{ cu. ft./sec.}$$
$$a = \frac{0{\cdot}107 \text{ cu. ft./sec.}}{0{\cdot}62 \times 50{\cdot}8 \text{ ft./sec.}}$$
$$= 0{\cdot}00275 \text{ sq. ft.}$$
$$= 0{\cdot}3961 \text{ sq. in.}$$

And if d is the diameter in inches,

$$d^2 = \frac{0{\cdot}3961}{0{\cdot}7854} = 0{\cdot}504 \text{ in.}^2$$
$$d = \sqrt{0{\cdot}504} = 0{\cdot}71 \text{ in.}$$

EXAMPLE 2. A jet 1 in. diameter issues from a nozzle connected to a supply of water at a pressure of 200 lb. per sq. in. Find the velocity of the jet, the flow in cubic feet per minute, and the horse-power available from the jet if 2 per cent. of the energy is wasted in friction.

Work done in foot-pounds by the water supply on the water issuing from the jet = pressure per square foot × volume in cubic feet. Considering the quantities per pound issuing,

$$\text{Volume} = \frac{1}{w} = \frac{1 \text{ lb.}}{62{\cdot}4 \text{ lb./cu. ft.}} = \frac{1}{62{\cdot}4} \text{ cu. ft.}$$

Work done

$$=\text{pressure/sq. ft.} \times \text{volume in cubic feet per lb.}$$

$$=200 \text{ lb./sq. in.} \times 144 \text{ sq. in./sq. ft.} \times \frac{1}{62{\cdot}4} \text{ cu. ft./lb.}$$

$$=\frac{200 \times 144}{62{\cdot}4} \text{ ft.-lb./lb.}$$

$$=461{\cdot}5 \text{ ft.-lb./lb.}$$

Loss of 2 per cent. (deduct) $=9{\cdot}2$ ft.-lb./lb

Kinetic energy in jet $=452{\cdot}3$ ft.-lb./lb.

$$\frac{v^2}{64{\cdot}4 \text{ ft./sec.}^2}=452 \text{ ft.-lb./lb.}$$

$$v^2=64{\cdot}4 \times 452 \text{ (ft./sec.)}^2$$

$$v=\sqrt{64{\cdot}4 \times 452} \text{ ft./sec.}$$

$$=172 \text{ ft./sec.}$$

Flow in cubic feet per minute

$$=172 \text{ ft./sec.} \times 60 \text{ sec./min.} \times \frac{0{\cdot}7854}{144} \text{ sq. ft.}$$

$$=\frac{172 \times 60 \times 0{\cdot}7854}{144} \text{ cu. ft./min.}$$

$$=56{\cdot}3 \text{ cu. ft./min.}$$

Since each pound has 452 ft.-lb. of K.E.,

Kinetic energy available per minute from jet

$$=56{\cdot}3 \text{ cu. ft./min.} \times 62{\cdot}4 \text{ lb./cu. ft.} \times 452 \text{ ft.-lb./lb.}$$

$$=56{\cdot}3 \times 62{\cdot}4 \times 452 \text{ ft.-lb./min.}$$

$$=1{,}588{,}000 \text{ ft.-lb./min.}$$

$$\text{h.p.}=\frac{1{,}588{,}000}{33{,}000}=48{\cdot}0$$

90. Head and Energy in Flow. We have seen that water moving, even slowly, under pressure transmits not only pressure but energy (Art. 86). In this respect the energy transmitted in any given time is the intensity of pressure

multiplied by the volume swept through, say, p lb. per sq. ft. $\times$ V cu. ft. $=pV$ ft.-lb. Or per pound of water passing a given plane perpendicular to the motion, the energy transmitted is

$$p \times \text{volume of 1 lb. in cu. ft.} = p/w \quad . \quad . \quad (1)$$

where w is the weight in pounds of 1 cu. ft. or $1/w$ the volume in cu. ft. of 1 lb.

It is sometimes said that water under pressure thereby has energy p/w ft.-lb. per lb., but this energy is no more contained in the water than is the energy transmitted by a piston (or other) rod, contained in the rod while transmission takes place. And water at rest under pressure does not contain energy p/w ft.-lb. per lb. any more than does, say, steel under pressure in a structure.[1]

Kinetic energy at $v^2/2g$ ft.-lb. per lb. may more properly be said to be in water moving at v ft. per sec. To emphasise

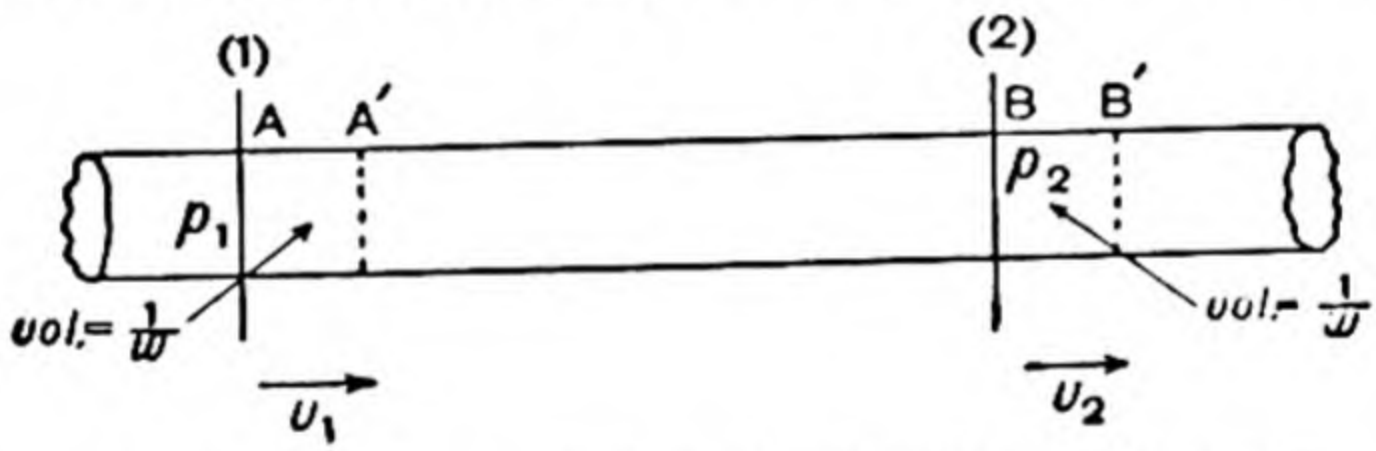


FIG. 114.—Transmission of pressure and energy.

a point, so often misstated, consider a pipe of uniform bore (Fig. 114) with water flowing at a uniform speed $v_1 = v_2$ ft. per sec. and at a pressure $p_1 = p_2$ lb. per sq. ft. While 1 lb. of water flows past a section (1), an equal volume $\left(\frac{1}{w}\right)$ cu. ft. flows past section (2). The work done *on* the water AB by the water to the left of A is $p \times$ vol. of 1 lb. $= p/w$ ft.-lb. per lb., and the work done by the water AB on that to the right of B, if there is no frictional loss, is also p/w ft.-lb. per lb. This work is transmitted in the same way along any length from a supply of mechanical energy to a place

[1] Both water and steel contain a *small* amount of elastic strain energy when under pressure, but this is negligible in comparison with p/w per lb. at ordinary pressures.

where it is used to do work, with or without conversion, to some other form of energy. And if the pipe is horizontal, the total energy passing per pound of water *flowing* is

$$p/w + v^2/2g \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the first term representing energy transmitted per pound by pressure and the second that carried as kinetic energy.

Now, neglecting friction, consider a horizontal pipe (Fig. 115) in which the section is not everywhere the same. While 1 lb. of water passes section (1), 1 lb. also passes section (2). The work done *on* the water AB at (1) is

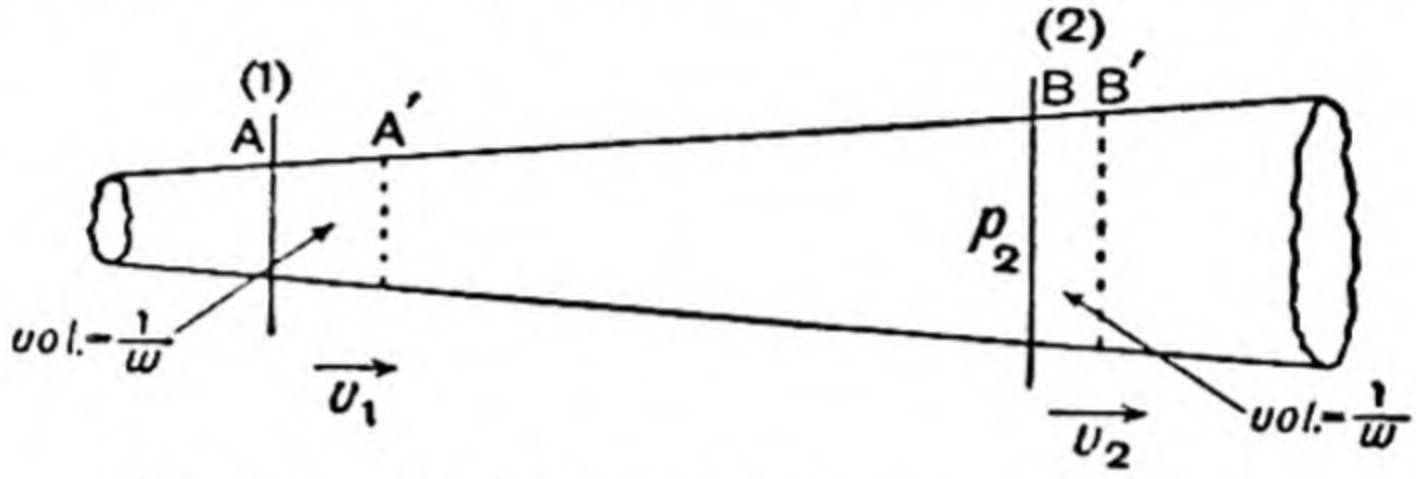


FIG. 115.—Variation of pressure and velocity.

p_1/w ft.-lb. per lb., and the work done *by* the water AB at (2) is p_2/w. The net work done by the water is therefore $p_2/w - p_1/w$ or $(p_2 - p_1)/w$ per lb. of water passing a given point; and from the principle of the conservation of energy, this must be at the expense of its kinetic energy, which is reduced from $v_1^2/2g$ to $v_2^2/2g$ or

$$\frac{v_1^2}{2g} - \frac{v_2^2}{2g} = \frac{p_2}{w} - \frac{p_1}{w} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which may be written

$$\frac{v_1^2}{2g} + \frac{p_1}{w} = \frac{v_2^2}{2g} + \frac{p_2}{w} \quad . \quad . \quad . \quad . \quad (4)$$

We have indicated that it would be misleading to say that equation (4) indicated equal energy in each pound of water. It would be better to say, as indicated by (3), that per pound of water passing between any two sections, the loss of kinetic energy is equal to the net work done *by* the water; or the gain of energy, where it is accelerating, is equal to the net work done *on* the water. Note that if there is no

flow, no energy is transmitted—only force. Movement is an essential part of the transmission of the energy.

Bernoulli's Theorem. For a pipe which is not horizontal, we must add the net work done by or against gravity as the case may be, viz. 1 ft.-lb. per lb., per foot difference of level. Thus to lift 1 lb. of water from a level h_1 ft. above a datum at (1), to a level h_2 at (2) (a vertical distance of (h_2-h_1) ft.), requires (h_2-h_1) ft.-lb. of work per pound, which must be added to the right-hand side of (3) (as additional work done at the expense of the K.E. per pound) making the equation

$$\frac{v_1^2}{2g}-\frac{v_2^2}{2g}=\frac{p_2}{w}-\frac{p_1}{w}+h_2-h_1 \quad . \quad . \quad . \quad (5)$$

and this makes (4) become

$$h_1+\frac{v_1^2}{2g}+\frac{p_1}{w}=h_2+\frac{v_2^2}{2g}+\frac{p_2}{w} \quad . \quad . \quad . \quad (6)$$

This is known as Bernoulli's theorem relating to the stream line flow of a frictionless liquid.

If the equations (4) and (6) do not indicate equal energy *contained in* 1 lb. of water passing sections (1) and (2), what equality do they indicate? We have seen that water moving vertically upward as a jet with velocity v_1 ft. per sec., with no friction, would reach a height of $v_1^2/2g$ ft. We may then call $v_1^2/2g$, the kinetic head at section (1). And water issuing freely from a stationary supply at p_1 lb. per sq. ft. into the atmosphere would have a velocity of $v=\sqrt{2gp_1/w}$, or sufficient to raise it to a vertical height p_1/w ft. The quantity p_1/w is then called a "pressure head", and the total head in feet at section (1) is $h_1+\frac{v_1^2}{2g}+\frac{p_1}{w}$ and this also equals $h_2+\frac{v_2^2}{2g}+\frac{p_2}{w}$, so that Bernoulli's theorem, expressed in symbols in equation (6), may be stated in words as follows. In a pipe full of a flowing frictionless liquid, the total head is constant. The total heads at sections (1) and (2) are made up of (*a*) gravitational head (above an arbitrary datum level); (*b*) kinetic head, and (*c*) pressure head. It may be well to emphasise that the terms in the expressions (5), (6), etc., are not in amounts of energy.

They are not foot-pounds but foot-pounds *per pound*, which dimensionally are lengths in *feet*. Similarly $1/w$ is not dimensionally volume in cubic feet but volume in cubic feet *per pound*.

The relation embodied in equation (6), known as Bernoulli's Theorem, has a wider application than to flow in a pipe.

In Fig. 116 we show the conditions at four places A, B, C, and D. At A and B, the total heads consist of all three parts, while at C the head is wholly kinetic, and at D it is wholly gravitational. The total head in all cases is h and the total heads at A, B, C, and D are made up as follows :

Place	Gravitational head	Kinetic head	Pressure head	Total head
D	h	0	0	h
A	h_A	$\frac{v_A^2}{2g}$	$\frac{p_A}{w}$	$h_A+\frac{v_A^2}{2g}+\frac{p_A}{w}=h$
B	h_B	$\frac{v_B^2}{2g}$	$\frac{p_B}{w}$	$h_B+\frac{v_B^2}{2g}+\frac{p_B}{w}=h$
C	0	$\frac{v_C^2}{2g}$	0	$\frac{v_C^2}{2g}=h$

The total head is the same in each case and if p is in pounds per square foot, $w=62{\cdot}4$ lb. per cu. ft., v is in feet per second, and g is 32·2 ft. per sec. per sec.

Many interesting points arise from the foregoing simple relations but we cannot pursue them far at this stage. But we may again refer to equation (3) for two different sections of a tube (Fig. 115) conveying water in steady flow, viz.

$$\frac{v_1^2}{2g}-\frac{v_2^2}{2g}=\frac{p_2-p_1}{w}$$

If the section (1) is a narrow channel or " throat " in a tube which is elsewhere of the larger section, as at (2), v_1 can be made greatly in excess of v_2; then v_1^2 greatly exceeds v_2^2 and consequently p_2 greatly exceeds p_1. Thus

a " throat " in a tube conveying fluid can be made to have a pressure below atmospheric and to give a suction. This

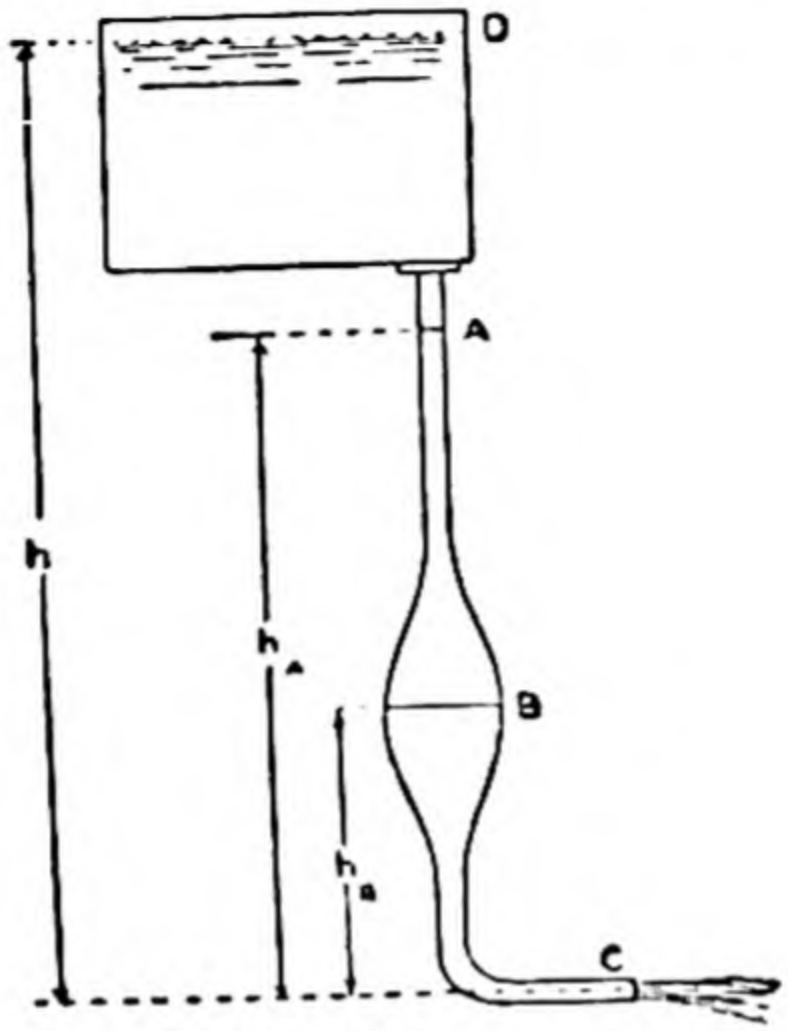


FIG. 116.—Variation of pressure with height and velocity.

gives just a glimpse of the principle of the injector which takes in water at atmospheric pressure and forces it into a boiler at much higher pressure, and of the choke tube of a carburettor which, at the throat of a tube conveying

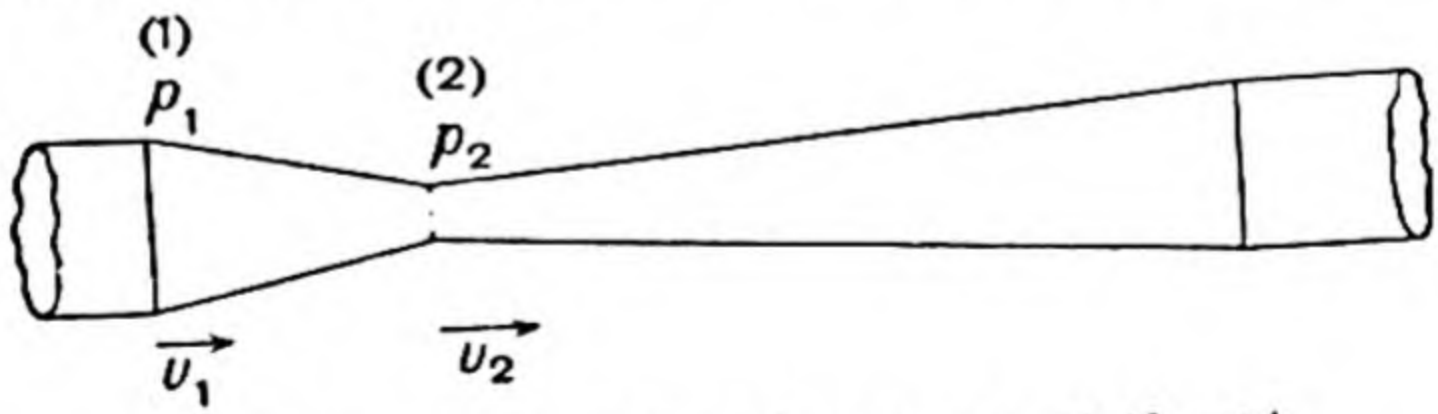


FIG. 117.—Reduction of pressure at throat.

air, has a low pressure to assist the evaporation of the petrol, which as a vapour then joins the stream of air, providing a mixture for combustion in the cylinder of a petrol engine. A measured pressure difference between the throat and the full-sized tube (Fig. 117) provides the means for deter-

mining the velocity of flow and hence the volume per second flowing in a pipe conveying water. Such is the principle of the Venturi water meter. (See Example 2 below.)

EXAMPLE 1. A horizontal pipe 0·2 sq. ft. in sectional area, conveys 0·4 cu. ft. of water per sec. It has a portion converging to a minimum section of 0·01 sq. ft. followed by a gradually diverging portion leading again to the full size. If the pressure in the main pipe is 20 lb. per sq. in. above atmospheric, find the pressure at the minimum section.

Using Fig. 117 for reference

$$v_1 = \frac{0{\cdot}4 \text{ cu. ft./sec.}}{0{\cdot}2 \text{ sq. ft.}} = 2 \text{ ft./sec.}$$

$$v_2 = \frac{0{\cdot}4 \text{ cu. ft./sec.}}{0{\cdot}01 \text{ sq. ft.}} = 40 \text{ ft./sec.}$$

Gain of kinetic head

$$= \frac{v_2^2 - v_1^2}{2g} = \frac{(40^2 - 2^2)(\text{ft./sec.})^2}{2 \times 32{\cdot}2 \text{ ft./sec.}^2}$$

$$= \frac{1596}{64{\cdot}4} \text{ ft.} = 24{\cdot}78 \text{ ft.}$$

This, from equation (3) or (4), etc., is equal to the loss of pressure head or, since $p_1 = 20$ lb. per sq. in. $\times 144$ sq. in. per sq. ft. $= 2880$ lb. per sq. ft.,

$$\frac{p_1}{w} - \frac{p_2}{w} = \frac{(2880 - p_2)}{62{\cdot}4 \text{ lb./cu. ft.}} = 24{\cdot}78 \text{ ft.}$$

$$2880 - p_2 = 24{\cdot}78 \times 62{\cdot}4 \text{ lb./sq. ft.}$$

$$= 1546 \text{ lb./sq. ft.}$$

$$p_2 = 2880 - 1546 = 1334 \text{ lb./sq. ft.}$$

or 9·26 lb./sq. in.

If the pressure in the pipe had been only 20 lb. per sq. in. absolute, the pressure at the throat would have been below atmospheric (14·7 lb. per sq. in.) and a pressure gauge would have shown a suction there.

EXAMPLE 2. With the tube and throat dimensions of Example 1, find the flow in cubic feet per second if a mercury U-tube (or manometer) connected to the full-sized pipe and the throat shows a difference of pressure represented by a difference of level of mercury of 10 in.

Since mercury is 13·6 times as heavy as water, 10 in. of mercury represents a head of $13{\cdot}6 \times 10$ in. $=136$ in. or 11·33 ft. of water, which we need not reduce to pounds per square feet.

Since the area of section at the throat is 0·01/0·2, or $\frac{1}{20}$, of the full section of the pipe, and since the total flow at both sections is the same, the velocity v_2 must be 20 times v_1; hence the gain in kinetic head is

$$\frac{v_2^2}{2g} - \frac{v_1^2}{2g} = \frac{(20v_1)^2 - v_1^2}{2g} = \frac{399v_1^2}{2g}$$

and this is equal to the loss of pressure head of 11·33 ft.

i.e. $$\frac{399v_1^2}{64{\cdot}4 \text{ ft./sec}^2} = 11{\cdot}33 \text{ ft.}$$

$$v_1^2 = \frac{11{\cdot}33 \times 64{\cdot}4}{399} \text{ ft.}^2/\text{sec.}^2$$

$$= \frac{729{\cdot}5}{339} \text{ (ft./sec.)}^2 = 1{\cdot}828 \text{ (ft./sec.)}^2$$

$$v_1 = 1{\cdot}352 \text{ ft./sec.}$$

Volume flowing per second is of length of 1·352 ft. and of section 0·2 sq. ft.

$$1{\cdot}352 \text{ ft.} \times 0{\cdot}2 \text{ sq. ft.} = 0{\cdot}27 \text{ cu. ft.}$$

91. Friction in Pipes. In the preceding article (90), frictional losses were considered to be negligible. But liquids are not perfect: they have viscosity and experience a tangential or shearing resistance exerting a drag against the direction of flow. This results in a loss of energy (converted into heat), which becomes apparent as a loss of pressure in a horizontal pipe of uniform section, in which the height h above any datum level and the velocity v are constant. Thus, if in (6) of Art. 90, h_1, v_1, and p_1 relate

to a section in a full pipe and h_2, v_2, and p_2 relate to another later section, the equation will be modified and become

$$h_1+\frac{v_1^2}{2g}+\frac{p_1}{w}=h_2+\frac{v_2^2}{2g}+\frac{p_2}{w}+h' \quad . \quad . \quad . \quad (1)$$

where h' is the loss of head. And if $h_1=h_2$ and $v_1=v_2$

$$h'=(p_1-p_2)/w \quad . \quad . \quad . \quad . \quad . \quad (2)$$

p_1-p_2 being the fall of pressure due to frictional loss.

Experiment shows that where a certain critical velocity (at which the flow changes from one parallel to the wall to one of turbulence) is exceeded, this loss is proportional to the square of the velocity and to the length l of the pipe, and inversely proportional to the diameter d of a pipe of circular section, or

$$h' \propto lv^2/d \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

More generally, to include other shapes of pipes, h' is proportional to the wetted surface S of the pipe and inversely proportional to the area A of cross-section; and if we write h' in terms of the kinetic head $v^2/2g$,

$$h' \propto \frac{S}{A} \times \frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or

$$h'=f \times \frac{S}{A} \times \frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad (5)$$

where f is a coefficient expressing the frictional resistance of the wall of the pipe (but not the normal coefficient of sliding friction applicable to two solids). Experiment further shows that this coefficient f is not entirely independent of the diameter or cross-sectional area of the pipe, and equation (5) must be taken only as an approximate working rule. In the case of a round pipe of diameter d and length l, the wetted surface S is πdl (if the pipe is full) and $A=\pi d^2/4$, hence (5) becomes

$$h'=\frac{4fl}{d} \cdot \frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where d, h', and l are in feet, and v and g in foot and second units.

In the more general case, to include pipes of any section, the loss is stated in terms of l and the *hydraulic mean depth*, m, *i.e.* the area of cross-section of liquid divided by the wetted perimeter (the whole perimeter if the pipe is full). The wetted perimeter is the wetted surface S divided by the length l, or $m = A \div (S/l)$; hence (5) becomes

$$h' = \frac{flv^2}{2gm} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which reduces to (6) for the particular but common case of a round pipe in which $m = \frac{1}{4}\pi d^2/(\pi d) = d/4$, m and d being in feet (or in the same units as l).

Virtual Slope of a Hydraulic Pipe. The loss of head between two points on a pipe divided by the length of pipe between them, or the loss of head in feet per foot length of pipe, is called the virtual slope of the pipe.

Other Hydraulic Resistances. In addition to frictional resistances of the wall of a pipe, flow may be impeded by sudden enlargements or contractions of the pipe, elbows or sudden bends, valves, etc. The losses of head caused by these are matters for more advanced study of hydraulics and for experiment. The estimated or the experimental loss of head is usually stated in terms of the kinetic head $v^2/2g$ at some point such as that at which the water issues at the open end of the pipe.

EXAMPLE 1. If a pipe 900 ft. long and 5 inches diameter discharges 50 cu. ft. of water per minute and the coefficient f is 0·006, what is the loss of head due to friction and what is the virtual slope of this pipe?

Area of cross-section of pipe

$= 5^2 \times 0{\cdot}7854/144$ sq. ft. $= 0{\cdot}1364$ sq. ft.

Mean velocity of water

$= 50$ cu. ft. per min./0·1364 sq. ft. $= 366{\cdot}7$ ft./min.
$= 6{\cdot}112$ ft./sec.

Loss of head $2flv^2/(gd)$

$= 2 \times 0{\cdot}006 \times 900 \times 6{\cdot}112^2/(32{\cdot}2 \times 5/12)$

$= 30{\cdot}05$ ft.

Virtual slope $= 30/900 = 1/30$ or 1 in 30.

EXAMPLE 2. How much water per minute could be delivered by a pipe 3 inches in diameter and 400 ft. long from a reservoir to a tank 16 ft. below the (constant) water level in the reservoir. Take $f=0{\cdot}007$

Let v be the velocity of flow in pipe.

Kinetic head + loss of head in pipe = 16 ft.

$$\frac{v^2}{2g}+\frac{4fl}{d}\cdot\frac{v^2}{2g}=16 \text{ ft.}$$

$$\frac{v^2}{2g}\left(1+\frac{4\times 0{\cdot}007\times 400 \text{ ft.}}{0{\cdot}25 \text{ ft.}}\right)=16 \text{ ft.}$$

$$\frac{v^2}{2g}(1+44{\cdot}8)=16 \text{ ft.}$$

$$v^2/2g=16/45{\cdot}8$$

$$v^2=2\times 32{\cdot}2\times 16/45{\cdot}8$$

$$v=4{\cdot}742 \text{ ft./sec.}$$

Area of section of pipe

$$=3^2\times 0{\cdot}7854/144=0{\cdot}04909 \text{ sq. ft.}$$

Quantity of water flowing

$$=4{\cdot}742 \text{ ft./sec.}\times 0{\cdot}04909 \text{ sq. ft.}\times 60 \text{ sec.}$$

$$=13{\cdot}97 \text{ cu. ft./minute.}$$

EXAMPLE 3. A supply of 30 gallons of water per minute is required at the end of a pipe 5000 feet long, the head of water being 25 feet above the discharge. Find the diameter of pipe required if $f=0{\cdot}007$.

Let v = velocity in ft./sec. and d = diameter of pipe in ft.

Discharge per second $=30/60=0{\cdot}5$ gal. $=0{\cdot}5/6{\cdot}24$ cu. ft. $=0{\cdot}08013$ cu. ft.

$$v=\frac{0{\cdot}08013 \text{ cu. ft./sec.}}{0{\cdot}7854d^2 \text{ sq. ft.}}=0{\cdot}10203/d^2 \text{ ft./sec.}$$

Kinetic head + loss in pipe = 25 ft.

$$\frac{v^2}{2g}+\frac{4\times 0{\cdot}007\times 5000}{d}\times\frac{v^2}{2g}=25 \text{ ft.}$$

We can neglect $v^2/2g$ in comparison with the loss of head in the long pipe and write,

$$\frac{4 \times 0{\cdot}007 \times 5000}{d} \times \frac{0{\cdot}10203^2}{d^4 \times 64{\cdot}4} = 25$$

$$140 \times 0{\cdot}01041/d^5 = 25 \times 64{\cdot}4 = 1610$$

$$d^5 = 1{\cdot}4574/1610 = 0{\cdot}000905.$$

$$d = \sqrt[5]{0{\cdot}000905} = 0{\cdot}2462 \text{ ft.}$$

or 2·95 inches, say 3 inches.

(We may now verify that the neglected kinetic head $v^2/2g$ is really negligible in comparison with the loss of head. The latter is

$$(v^2/2g) \times 4 \times 0{\cdot}007 \times 5000/0{\cdot}25 \text{ or } 560\ v^2/2g$$

i.e. the neglected kinetic head is about 1/561 part of the head of 25 ft.)

EXAMPLE 4. A pipe is 2000 feet long and 6 inches diameter. What diameter of nozzle at the discharge end will be required if it is to deliver 75 cubic feet of water per minute under a head of 80 feet of water at entry to the pipe line? Neglect losses other than the frictional loss in the pipe, for which the coefficient f may be taken as 0·007.

$$75 \text{ cu. ft./min.} = 1{\cdot}25 \text{ cu. ft./sec.}$$

Velocity at nozzle of diameter d ft.

$$= 1{\cdot}25/0{\cdot}7854d^2 = 1{\cdot}5915/d^2$$

Velocity in pipe

$$= 1{\cdot}25 \text{ cu. ft.}/(0{\cdot}7854 \times 0{\cdot}5^2) \text{ sq. ft.} = 6{\cdot}366 \text{ ft./sec.}$$

Loss of head in pipe

$$= \frac{4 \times 0{\cdot}007 \times 2000}{0{\cdot}5} \times \frac{6{\cdot}366^2}{64{\cdot}4} = 70{\cdot}49 \text{ ft.}$$

Head at nozzle $= 80 - 70{\cdot}49 = 9{\cdot}51$ ft.

i.e. $v^2/2g = 9{\cdot}51$, $v = \sqrt{9{\cdot}51 \times 64{\cdot}4} = 24{\cdot}75$ ft./sec.

Area of nozzle section

$$= 1{\cdot}25 \text{ cu. ft. per sec.}/24{\cdot}75 \text{ ft. per sec.}$$

$$= 0{\cdot}05051 \text{ sq. ft.}$$

$$d = \sqrt{0{\cdot}05051/0{\cdot}7854} = 0{\cdot}2536 \text{ ft. or } 3{\cdot}04 \text{ inches.}$$

EXAMPLE 5. What power will be required to pump 400 gallons of water per minute to an elevation of 80 feet through a pipe 400 ft. long and 6 inches diameter if f is taken as 0·007 ?

400 gal./min.

$=400/(6{\cdot}24\times60)$ cu. ft./sec. $=1{\cdot}069$ cu. ft./sec.

Area of section of pipe

$=0{\cdot}7854\times0{\cdot}5^2=0{\cdot}1963$ sq. ft.

Velocity of flow $=1{\cdot}069$ (cu. ft./sec.)/0·1963 sq. ft.

$=5{\cdot}443$ ft./sec.

Loss of head in 400 ft.

$$=\frac{4\times0{\cdot}007\times400}{0{\cdot}5}\times\frac{5{\cdot}443^2}{64{\cdot}4}=10{\cdot}30 \text{ ft.}$$

Total head $=10{\cdot}3+80=90{\cdot}3$ ft.

Work per min. in pumping 4000 lb. of water

$=4000\times90{\cdot}3$ ft. lb.

h.p. $=4000\times90{\cdot}3/33{,}000=10{\cdot}94$, say 11 h.p.

92. Force Exerted by a Jet of Water. The moving water in a stream or jet has a certain momentum. If the water is turned from its motion in a straight line, it follows from the first law of motion (Art. 57) that force must be exerted upon it. And from the second law of motion (Art. 58) that this force is measured by the rate of change of momentum of the stream and is in the direction of the change of momentum. And from the third law that the force exerted on the stream by any obstacle which serves to change its momentum is equal and opposite to the force exerted by the stream or jet on the obstacle.

Consider the force exerted by a jet of water on, say, a fixed flat plate. If we know the quantity of water flowing and its velocity, we can calculate its momentum. The force exerted in a direction opposite to that of flow, *by* an obstacle, will depend upon the degree of deviation of the stream of water; it may be through any angle from zero

to complete reversal of direction or 180°. The change of momentum and the force are of course vector quantities, and in any but simple deviations it can best be estimated by vectors.

Suppose a steady stream of water in the form of a horizontal jet strikes a vertical fixed flat plate, as in Fig. 118, and the water flows off at right-angles to its original direction, *i.e.* it is deviated through an angle of 90°. The water in the jet undergoes a complete loss of momentum in the original direction of motion, that is, an equal gain of momentum in the direction opposed to the motion, and the rate of this gain measures the force exerted *upon the water by the plate* in the direction opposite to that of flow. Thus suppose the velocity of the jet to be 30 ft. per sec. and that 50 lb. of water are carried to the surface of the plate per sec., then the change (loss) of momentum in the original direction per second is

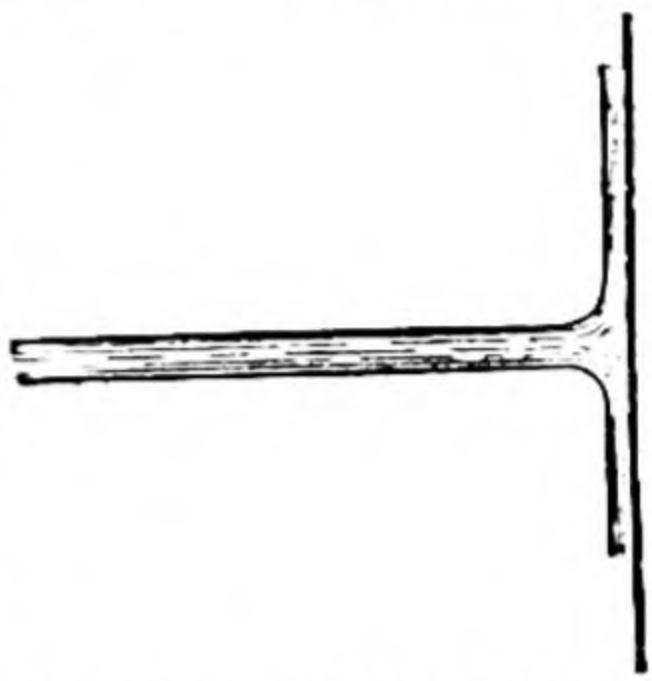


FIG. 118.—Force of a jet on a fixed plate.

$$\text{mass} \times \text{velocity} = \frac{50 \text{ lb.-force}}{32{\cdot}2 \text{ ft./sec.}^2} \times 30 \text{ ft./sec.}$$

or rate of change of momentum

$$= \frac{50 \times 30}{32{\cdot}2} \frac{\text{lb.-force} \times \text{ft./sec.}^2}{\text{ft./sec.}^2}$$

$$= \frac{1500}{32{\cdot}2} \text{ lb.-force} = 46{\cdot}6 \text{ lb.}$$

If the stream were deviated through an angle θ, the final momentum in the original direction would be $mv \cos \theta$ and the loss in the original direction would be $mv - mv \cos \theta$ or $mv(1 - \cos \theta)$, which varies from 0 to $2mv$ (when $\theta = 180°$). For deviation at right-angles, $\theta = 90°$ $\cos \theta = 0$, and the loss of momentum in the direction of flow is mv.

The rate of gain of momentum measures the force exerted on the water; and the opposite reaction, equal to the rate of loss of momentum in any direction, therefore measures the force exerted by the jet.

Putting the foregoing in symbols, for the flat plate perpendicular to the jet, let W be the weight of water delivered by the jet per second. The force P exerted on and by the plate is the rate of change of momentum Wv/g where v = velocity in feet per second

$$P = Wv/g \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

And if the area of cross-section of the jet is A sq. ft., $W = Av \,.\, w$ where w is the weight of water per cubic foot, so that

$$P = Av^2w/g \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We may note that for a frictionless liquid (or for water in so far as we can ignore the friction between it and a plate), though there is force exerted by the jet on the fixed plate, no work is done and no energy is lost. The water flows on with undiminished speed and kinetic energy despite its change in direction. In machines such as water wheels and turbines for absorbing the energy in moving water, a stream of water exerts force on a *moving* body; it does work upon it and loses some of its energy.

If the flat plate perpendicular to the jet is moving away in the same direction as the jet with velocity v', the weight of water reaching the plate per second will be $Aw(v-v')$ and the change of velocity will be only $v-v'$, so that the rate of change of momentum will be $Aw(v-v')^2/g$, or

$$P = Aw(v-v')^2/g \quad . \quad . \quad . \quad . \quad (3)$$

and the work done per second will be v' times the force P,

or $$\text{work per second} = Aw(v-v')^2v'/g \quad . \quad . \quad (4)$$

The total energy per second available in the jet is $\frac{1}{2}Wv^2/g$ or

$$\text{Energy per sec.} = \tfrac{1}{2}Awv^3/g \quad . \quad . \quad . \quad . \quad (5)$$

And

$$\text{Efficiency} = \frac{Aw(v-v')^2v'/g}{\frac{1}{2}Awv^3/g} = \frac{2(v-v')^2v'}{v^3} \quad . \quad (6)$$

If, however, a succession of plates or vanes (say, radially on a wheel of large diameter) were arranged so that all the

water reached the vanes, W would be Awv and P would be $Awv(v-v')/g$,

$$\text{Work done per sec.} = Awv(v-v')v'/g \quad . \quad (7$$

$$\text{Efficiency} = 2v'(v-v')/v^2 \quad . \quad . \quad (8)$$

which would reach a maximum when $v' = \frac{1}{2}v$, and the efficiency would be 50 per cent., the maximum possible with radial vanes.

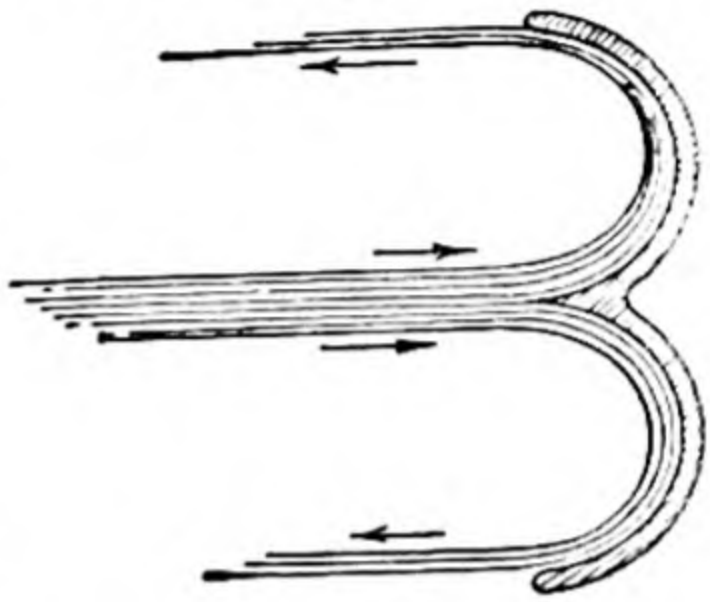


FIG. 119.

We may note one important case of deviation of the jet through (ideally) 180° by directing a jet centrally on to a hemispherical cup or equivalent arrangement shown in Fig. 119. If this cup is fixed, the change of velocity is from $+v$ to $-v$, *i.e.* a change of $2v$ in the direction of motion or $\theta = 180°$ and $(W/g)v(1-\cos\theta) = 2vW/g$, and (1) is replaced by

$$P = 2vW/g \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or in place of (2) we have

$$P = 2Awv^2/g \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Pelton Wheel. This type of cup or bucket is employed in the Pelton (impulse) wheel, a form of water motor used to develop power from high falls, *i.e.* from water with a high velocity of flow from a nozzle, see Fig. 120. Since there is a series of buckets, all the water, ideally at least, reaches the buckets and has its direction of flow relative to the wheel reversed, so that instead of P being $Awv(v-v')/g$,

$$P = 2Awv(v-v')/g \quad . \quad . \quad . \quad . \quad (11)$$

and instead of (7)

$$\text{Work per sec.} = 2Awvv'(v-v')/g \quad . \quad . \quad (12)$$

and

$$\text{Efficiency} = 4v'(v-v')/v^2 \quad . \quad . \quad . \quad (13)$$

These reach their maximum values when $v' = \frac{1}{2}v$ and the efficiency (ideally) is 100 per cent. The relative velocity

$v-v'$ of the water on leaving the bucket would then be $\frac{1}{2}v$, the velocity being zero, and all the energy and motion would have been taken from it. Actually not all the water reaches the buckets at the best angle, and the water cannot be turned through 180°, or it would not clear the following bucket. Also there are losses in friction in the feeding pipe, the jets, and the buckets, but the efficiency of the wheel is often as high as 80 per cent.

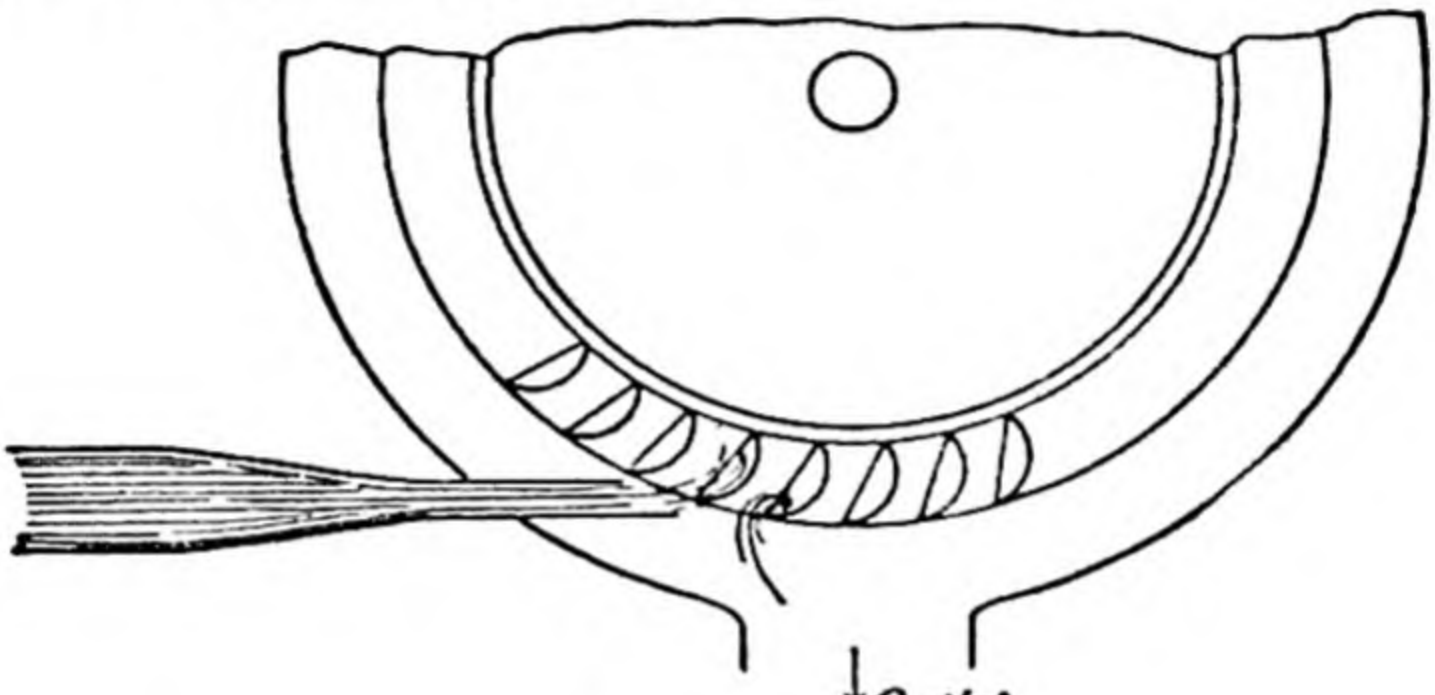


FIG. 120.

EXAMPLE 1. 5 cubic feet of water are supplied per second to a Pelton wheel under a head of 125 feet. The area of the nozzle is 0·056 square foot, and the b.h.p. of the wheel is 56; find (1) velocity of the jet; (2) work available per second; (3) efficiency of the wheel.

(1) Velocity of water $\times$ area $=5$ cubic feet.

$$v=5/0{\cdot}056=89{\cdot}2 \text{ feet per second.}$$

(2) Weight of water striking wheel per second

$$=5\times62{\cdot}4=312 \text{ lb.}$$

Available energy per sec. $=312$ lb. $\times 125$ ft.

$$=39{,}000 \text{ ft.-lb.}$$

$$\text{Available h.p.}=39{,}000/550=70{\cdot}9$$

(3) 56 h.p. $=56\times550=30{,}800$ ft.-lb. per sec.

$$\text{Efficiency of wheel}=30{,}800/39{,}000=0{\cdot}79$$

$$=79 \text{ per cent.}$$

Example 2. A Pelton wheel develops 82 h.p. at 950 r.p.m. from water at a head of 2000 ft. Estimate the quantity of water used per minute, taking the overall efficiency as 80 per cent. What is the torque on the shaft? If the wheel is 3 ft. diameter and the jet 0·5 inch diameter, what is the ratio of speed of jet to that of the buckets? What is the horse-power of the jet?

Let W be the weight of water used per minute.

Useful work done per minute by water

$$= 82 \times 33{,}000 \text{ ft.-lb.}$$

$$0{\cdot}8 \times W \times 2000 \text{ ft.} = 82 \times 33{,}000 \text{ ft.-lb.}$$

$$W = 1691 \text{ lb.}$$

Torque $= 82 \times 550$ ft.-lb. per sec. $/ (950 \times 2\pi/60)$ radians per sec.

$$= 453{\cdot}2 \text{ lb.-ft.}$$

Area of jet section $= 0{\cdot}7854 \times 0{\cdot}5^2 = 0{\cdot}19635$ sq. in.

$$\text{Water speed} = \frac{1691}{60} \times \frac{1}{62{\cdot}4} \text{ cu. ft./sec.} \div 0{\cdot}19635/144 \text{ sq. ft.}$$

$$= 331{\cdot}3 \text{ ft./sec.}$$

Bucket speed $= 950 \times 3\pi$ ft./60 sec. $= 149{\cdot}2$ ft./sec

ratio water speed/bucket speed $= 331{\cdot}3/149{\cdot}2 = 2{\cdot}22$

Jet h.p. $= 1691 \times 331{\cdot}3^2/2 \times 32{\cdot}2 \times 33{,}000 = 87{\cdot}34$ h.p.

(The total h.p. used is $82/0{\cdot}8 = 102{\cdot}5$ h.p.)

(Or, 1691 lb. $\times$ 2000 ft./33,000 ft.-lb. $= 102{\cdot}5$ h.p.)

EXAMPLES IX

1. In a hydraulic press the ram is 10 inches in diameter, and the plunger of 1 inch diameter is worked by an effort of 30 lb. on a handle which gives it a leverage of 14 to 1. Assuming an overall mechanical efficiency of 80 per cent., what total pressure will be exerted by the ram?

2. How many gallons per horse-power-hour will be required from a hydraulic power main at 800 lb. per sq. in. if the efficiency is 70 per cent.?

3. A single-acting force pump has to deliver water against a head of 250 feet. Neglecting frictional losses, what must be

the average force exerted during the delivery stroke on the plunger, which is 4 in. diameter? If the stroke is 4 in. and there are 120 delivery strokes per minute, what horse-power will be required to drive the pump if its efficiency is 85 per cent.? How many gallons will be delivered per minute if the slip is 5 per cent.?

4. A pump is driven by an electric motor of 85 per cent. efficiency and delivers 100 gal. of water per minute under a head of 125 ft. The efficiency of the pump is 80 per cent. and the motor works at 200 volts. What current will it take? If the cost of electrical energy is 1*d.* per kWh., what is the cost of running the pump per hour?

5. Find the discharge per hour from a circular sharp-edged orifice 1 in. diameter under a constant head of 12 feet (coefficient of discharge is 0·62).

6. Find the diameter of a sharp-edged circular orifice to give a discharge of 70 gal. of water per hour under a head of 3 feet.

7. Water flows in a pipe of varying section into the atmosphere from a reservoir, the surface of which is 80 ft. above the outlet of the pipe. Neglecting friction, find the velocity of the water leaving the open end of the pipe and the pressure 50 feet above the open end if the velocity of the water at this level is 20 ft. per sec.

8. A pipe full of running water discharges 8 cu. ft. per sec. At a certain point A, the cross-sectional area is 4 sq. ft., and at B, 10 feet below A, the cross-section is 2 sq. ft. Find the difference of pressure between A and B.

9. A Venturi meter having a throat 2·4 inches diameter is fitted to a horizontal water main 6 inches diameter. A mercury manometer gauge shows a difference of pressure equal to 10·3 inches of mercury between the throat and the water in the 6-inch main before entry. Find the flow of water in gallons per hour, taking the specific gravity of mercury as 13·6.

10. Find the fall of pressure in lb. per sq. in. in a pipe 500 feet long and 4 inch diameter discharging 30 cu. ft. per minute, taking $f = 0{\cdot}0062$.

11. Find the discharge into the atmosphere, in gallons per minute, from a pipe 3 inches diameter and 500 feet long, the head of water at entry being 20 feet. Take $f = 0{\cdot}007$ and neglect all hydraulic losses except friction in the pipe.

12. What diameter of pipe will be required to give a delivery of 40 gallons of water per minute at the end of a horizontal pipe 6000 feet long if f is taken as 0·006 and the head of water is 30 feet?

13 What diameter of nozzle will be required to give a discharge of 300 gallons of water per minute at the end of a horizontal pipe 3000 feet long and 6 inches diameter with a head of 50 feet at entry? Take $f = 0{\cdot}007$ and neglect losses other than friction in the pipe.

14. Find the height to which a jet of water from a 1-inch nozzle would rise at the end of a 4-inch pipe 800 feet long if the head at the supply is 800 feet. Take $f=0{\cdot}007$. What is the velocity of the water on leaving the nozzle? Neglect losses in the nozzle.

15. A horizontal jet of water 4 inch diameter moving with a velocity of 50 ft. per sec. strikes a vertical fixed plate. Calculate the force exerted by the jet on the plate. What would it be if the jet were issuing under the pressure of a head of 80 feet of water? Neglect any loss of head in the nozzle.

16. Solve No. 15 if the plate were replaced by a cup which turns the water through an angle of 150°.

17. A Pelton wheel uses 5 gallons of water per second with a nozzle 1 inch diameter and develops 22 b.h.p. Find the velocity of the jet and the efficiency of the wheel.

18. A Pelton wheel develops 16 h.p. at 1100 r.p.m. Its jet is 0·5 inch diameter and the head of water is 1800 feet. If the overall efficiency is 75 per cent., how much water is used per minute and what is the ratio of water speed to bucket speed if the diameter of the wheel is 2·5 feet?

CHAPTER X

STRESS AND STRAIN

93. Stress. When a material transmits force it is said to be *stressed* or in a state of *stress*, and it undergoes *strain*, *i.e.* change in dimensions and generally of shape.

Tension. If a tie-bar AB, Fig. 121, is pulled by two forces P applied, say, by pins or bolts, at its ends, the material throughout all the length of the bar is transmitting the force P. If we consider any short length CD of the bar, it is pulled firstly at C by a force P, acting towards A, exerted by the material in the bar lying to the left of C.

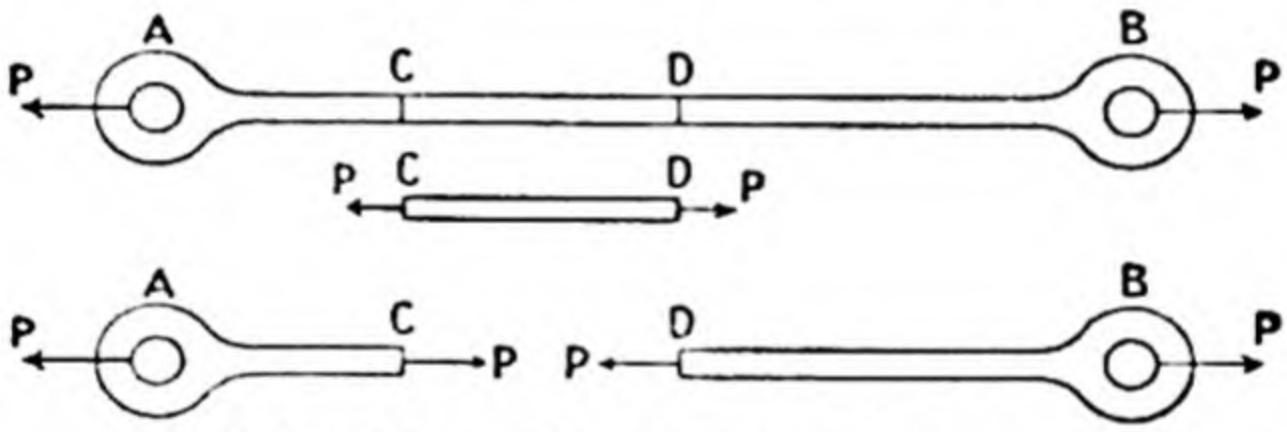


FIG. 121.—Tension in a tie-bar.

And secondly at D by an equal force P, acting towards B, exerted by the material to the right of D. Also, as illustrated at the bottom of Fig. 121, the piece CD exerts at C and D equal and opposite pulls to those exerted on it. It pulls leftward at D on the piece DB, and rightward at C on the piece AC.

Thus at any imaginary cross-section of the bar there are equal and opposite forces P. Each of the two pieces, into which any cross-section divides the bar, pulls the piece on the opposite side of the section towards itself. It is this *dual* action which constitutes ***tensile stress*** or *tension*.

The forces P shown at the top of Fig. 121 are those exerted outwards *on* the tie-bar *by* the pins at A and B. The forces exerted *by* the tie-bar *on* the pins would also be equal to P but inwards. If arrowheads are ever used

to show the forces in connection with such a bar, it must be made clear whether they indicate forces exerted *on* the tie-bar or those exerted *by* the tie-bar on the pins or other fastenings. The bar as a whole and every part of it can be regarded as a body acted upon by a system of forces in equilibrium and fulfilling the conditions of equilibrium (Art. 26).

The tension across sections such as C and D acts at right angles to those sections. This and compression, the reversed stress which would result from thrusts instead of pulls from the pins, are the simplest forms of stress and are called normal or direct stress. Across an oblique plane section of the bar, the stress would be inclined to the plane of the section ; it can be resolved into components normal to and along, or tangential to, the section.

Compression. Fig. 122 shows a short thick bar, AB, of material, with opposing forces P pushing axially at its

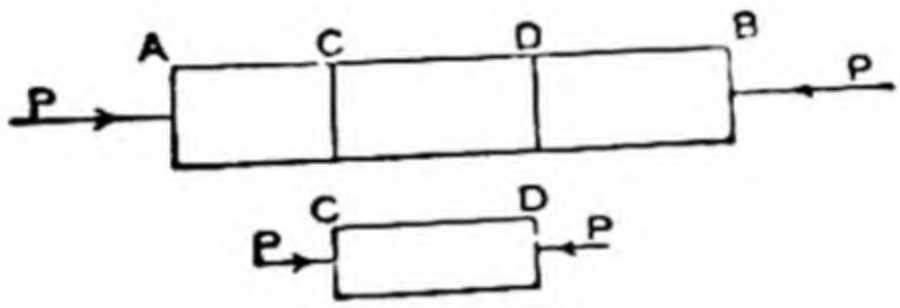


FIG. 122.

ends, that is, in the opposite directions to the forces in Fig. 121 which cause tension. The material is then in *compression*, or under *compressive stress* or *pressure*. The transmission of force is just as in tension, with equal and opposite normal forces across any section perpendicular to the axis of the bar. A short length CD is acted upon by normal *inward thrusts* P across the faces C and D as shown and exerts *outward* thrusts, each equal to P, on the parts AC and DB.

Shear Stress. Material has *shear stress* across a plane surface in it when there are forces in that surface tending to make the part on one side of it slide on the part on the other side. Thus a pin or rivet shown in Fig. 123 is subject to shear stress across the plane surface AB (shown dotted). The upper part of the pin A is urged by the force P to slide to the left relative to the lower part B, but the force exerted by B on A across the surface AB, to the

right, resists such motion or urges it in the opposite direction. The tendency of the shear stress is to *shear* the pin into two parts A and B. Very common cases of the use of metal to resist shearing are to be found in riveted joints, and in very numerous cases where one part of a machine

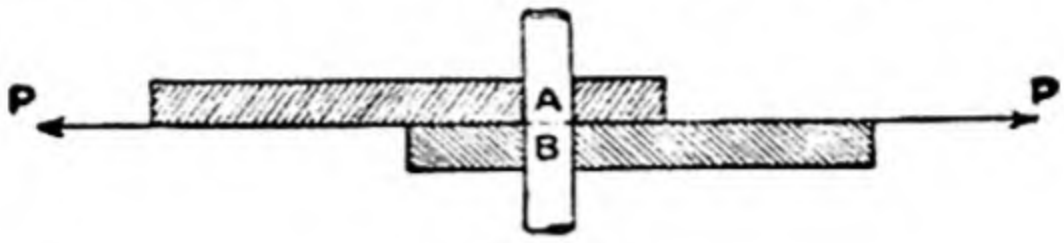


FIG. 123.

is capable of turning relative to another about a pin. For example, a knuckle joint, Fig. 124, is very commonly employed. The pin resists the pull P at *two* sections, A and B, so that the average intensity of shear stress in the pin is equal to P divided by *twice* the area of cross-section of the pin. The pin is said to be subjected to *double shear*.

Intensity of Stress. Stress is often specified by the force transmitted per unit area of some section, *e.g.* pounds per

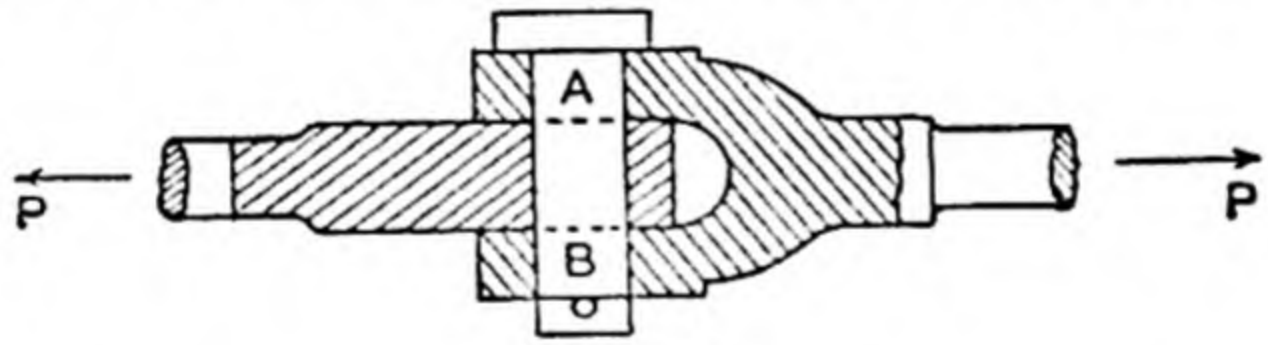


FIG. 124.

square inch. It is then more strictly called the *intensity of stress* (or occasionally unit stress), but frequently it is simply called stress. And such an expression as " a stress of 6 tons per square inch " is permissible, for it is obviously an intensity or force divided by area. Such a stress may be uniformly distributed over an area or it may vary greatly in intensity. The intensity at a point is the limiting value of force divided by area when both area and force are reduced indefinitely. The standard symbol for an intensity of normal stress is p and for an intensity of shear stress is q.

94. Strain and Elasticity. Strain is the alteration of shape which occurs when a material is stressed, *i.e.* when

it transmits force. Strain may be elastic or it may be partly inelastic. A material is elastic if on the removal of stress the accompanying strain disappears and the material returns to the shape and dimensions which it had before it was stressed. Many metals are almost perfectly elastic within limited intensities of stress, and *elasticity* of a material may be defined as the property of returning to its former shape and dimensions on the removal of stress.

Tensile and Compressive Strains. The strain produced, in the direction of the stress, by direct stress (tensile or compressive) is measured by the proportional increase or decrease of length. Thus if a rod 4 feet long stretches $\frac{1}{8}$ inch, the proportional strain is as follows :—

$$\text{Tensile strain} = \frac{\text{total stretch}}{\text{original length}} = \frac{0{\cdot}125 \text{ in.}}{4 \times 12 \text{ in.}} = 0{\cdot}0026.$$

The total stretch and total length must be reckoned in the same units of length to get the (proportional) strain. The elastic strain as thus measured will be the same whether calculated for the whole length or for any part of it.

Shear Strain. The form of shear strain which results from a shear stress is illustrated in Fig. 125, which represents a number of piled metal plates which in the right-

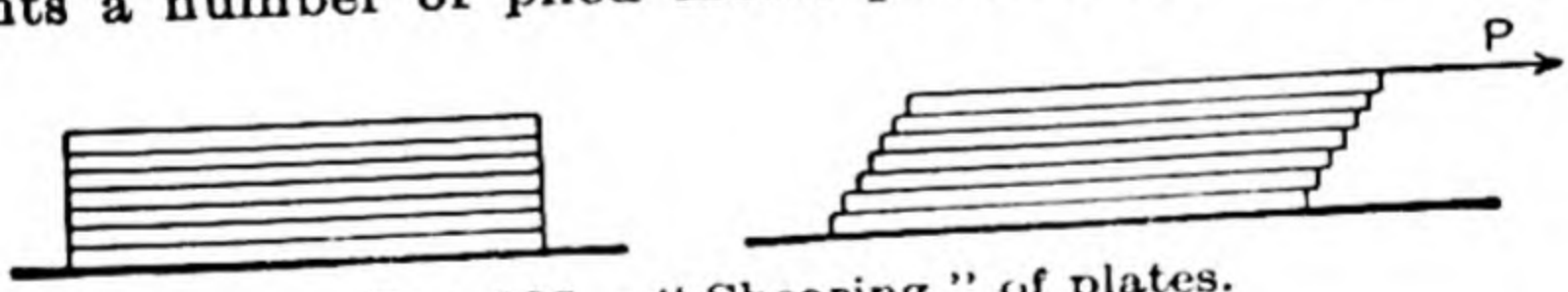


FIG. 125.—" Shearing " of plates.

hand view have been pulled to the right. Suppose these plates are separated one from another by layers or films of thick oil. Then the movement to the right due to a pull P will be resisted by each film of oil and the films will be subject to shear stress. In continuous material, the form will be as illustrated in Fig. 126, which might represent the visible shear strain which can be produced in a jelly. Visible shear strains can be produced in rubber, but simple shear strains in metal do not lend themselves to easy visual demonstration. The measure of a shear strain is the angular distortion, *e.g.* in Fig. 126 the rectangular

block of material has had its vertical edges sheared over to an inclination, say, ϕ radian to the vertical, and ϕ (usually a very small fraction) is the measure of the shear strain.

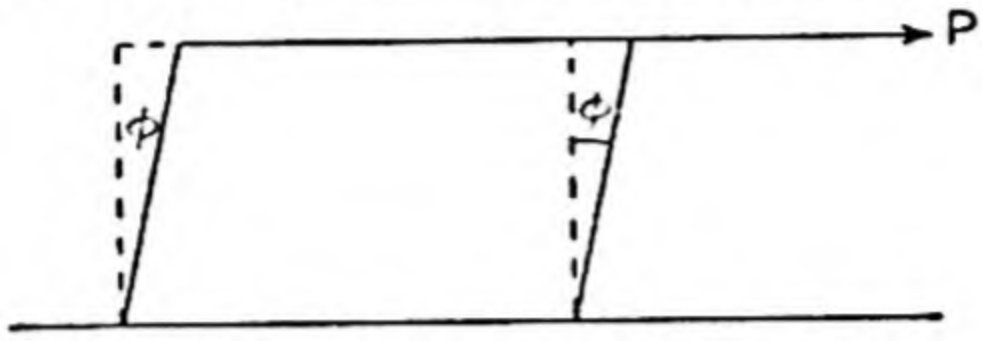


FIG. 126.—Form of shear strain.

95. Tensile Straining of Iron Wire. The straining properties of various metals in the form of wires may be demonstrated easily, because by using a long length of wire the stretch is large enough to be visible and measured without the use of special instruments. Experiments on wire may be made without the use of elaborate testing machines. The results of a test of iron wire 0·055 inch diameter (0·00237 sq. in. cross-sectional area) and 90·7 inches long are shown in the following table.

Load (lb.) .	0	10	20	30	40	50	60	70	80
Stretch (in.) .	0	0·01	0·022	0·035	0·05	0·062	0·075	0·10	0·20

Load (lb.) .	90	100	105	110	118	128	135	144	149	154
Stretch (in.) .	0·40	0·75	1·04	1·29	1·78	2·18	2·91	4·27	5·4	7·0 broke

The load and stretch are shown plotted as a graph in Fig. 127. An examination of the tabulated figures or of Fig. 127 will show that there are two stages in the stretching of the wire :—

(1) *The Elastic Stage.* Up to a load of about 60 pounds, the strain is small and is proportional to the load, the latter fact being much more clearly shown by replotting this part of the graph on a larger scale for extensions, as adopted in Fig. 128. During this stage of the test it will be found that on removal of the load the stretch disappears and the wire returns to its original length. The fact that

during this elastic stage the stretch is proportional to the load is characteristic of the elastic straining of metals. Although the tensile strains of steel during the elastic stage may not be visible to the unaided eye (unless in a

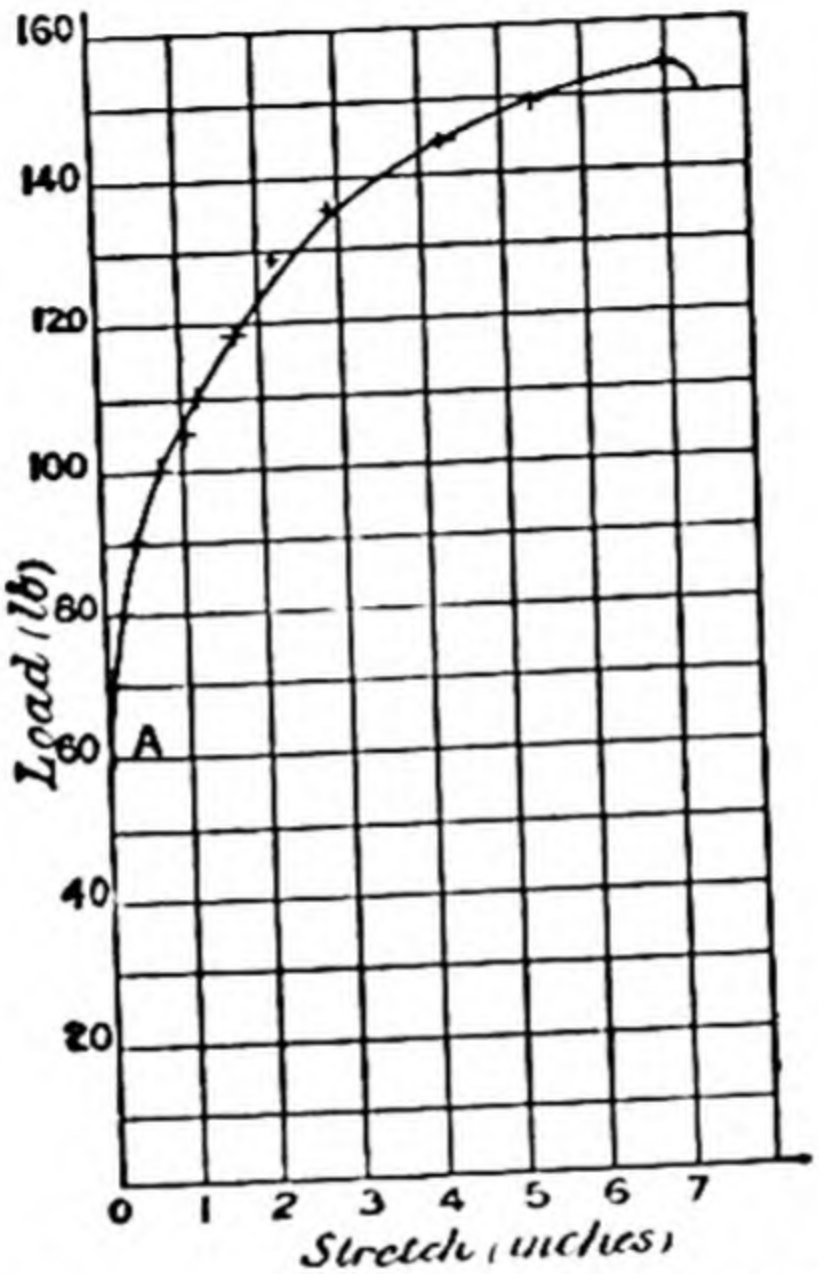


FIG. 127.—Tensile test of a long iron wire.

long wire), nevertheless the material behaves under stress like a common spring.

(2) *The Plastic Stage.* From about 60 pounds upwards the strain increases much more rapidly, so that when the load is 149 pounds, or about 2·5 times as much as at 60 pounds, the strain has increased more than 70 times. A metal which can thus be drawn out in a plastic condition is said to have the quality of *ductility* or to be *ductile*. *Plastic* strain, or a large part of it, remains if the load is removed, and the metal is said to have *permanent set*.

The ultimate strength of the wire reckoned on the

intensity of stress on the original unstrained area of cross-section is

$$\frac{\text{breaking load}}{\text{original area of section}} = \frac{154 \text{ lb.}}{0{\cdot}00237 \text{ sq. in.}}$$

$$= 64{,}990 \text{ lb. per sq. in.}$$

(or about 29 tons per sq. in.).

This is the *nominal* stress, because it is reckoned on the original area of section, which may have been reduced substantially before fracture; in relatively soft material there is considerable local contraction of section about

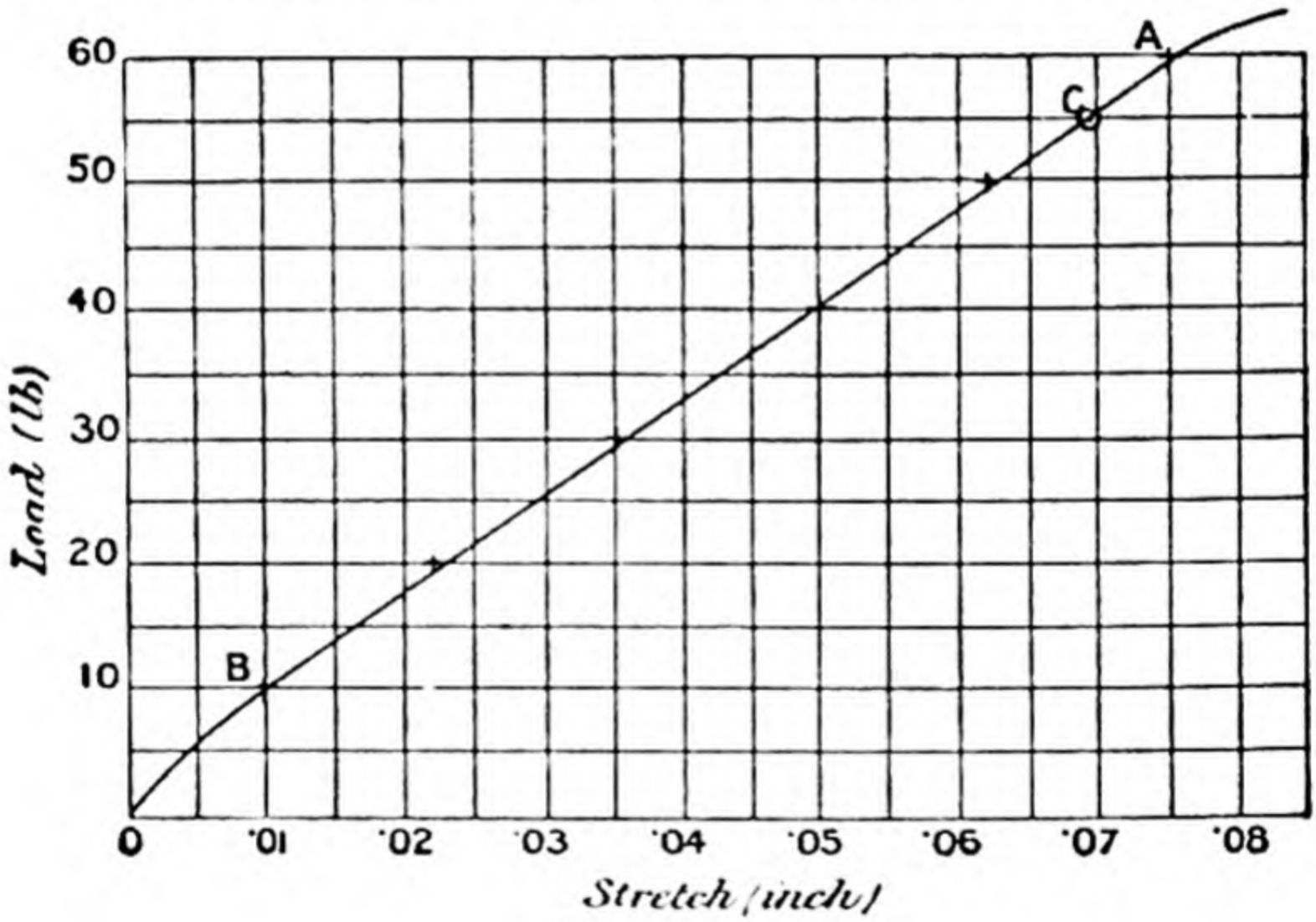


FIG. 128.

the part of the wire where fracture occurs, a visible waist being formed.

The final elongation taken over this long length of wire is 7 inches on 90·7 inches, or $7/90{\cdot}7 = 0{\cdot}077 = 7{\cdot}7$ per cent. It will obviously be a greater proportion if reckoned on a shorter length of the wire if there is much local extension about the waist formed at the section where fracture occurs. It should be observed that while simple experi-

ments on wires form an introduction to more elaborate tests on metal, the properties of materials, and notably of steels, are considerably modified by the process of manufacture of wire by drawing cold through a hole in a die. The nominal ultimate strength is increased and ductility is reduced. These modifications in properties may be largely removed by a suitable degree of heating and regulated cooling.

96. Hooke's Law. A material which, within the elastic stage of deformation, is strained by an amount proportional to the stress which produces the strain is said to behave according to *Hooke's Law*. Robert Hooke was a scientist who, in 1676, first published a clear statement of this relationship. His law has wide application, and is true alike for tension, compression, bending, torsion, or other form of shear stress and strain.

97. Modulus of Elasticity. While for a single stretched wire the strain is proportional to the load, for wires of any diameter it is evidently the intensity of the stress, that is, load divided by area, to which the strain is proportional. For if we consider twice the load borne by twice the area of cross-section (for example, by two wires instead of one), evidently the strain and the stress intensity would be unaltered. The fact that the stress intensity is proportional to the strain may be written :

$$\text{Stress intensity} = \text{strain} \times \text{constant},$$

and therefore

$$\text{the constant} = \frac{\text{stress intensity}}{\text{strain}} \quad . \quad . \quad . \quad (1)$$

This constant, relating to direct stress (tension or compression) is called *Young's Modulus of Elasticity*, after Thomas Young, who first gave the constant a physical meaning. The standard symbol to denote it is the letter E.

$$\text{E} = \text{stress intensity/strain} \quad . \quad . \quad . \quad (2)$$

or

$$\text{E} = p/e \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where e is the proportional strain.

Young's modulus may be regarded as the measure of the resistance which the material offers to stretching, in the sense that the larger the value of E, the smaller is the stretch under a given intensity of stress. Or for a given stress. the modulus is a measure not of the extensibility but the inextensibility of a material.

From the preceding expression for E it is evident that if the intensity of stress in any case, and also the (proportional) strain, are known, the numerical value of E can be calculated for the material in question. Also if E is known and the strain can be measured by a gauge, the intensity of stress can be calculated, *e.g.* in a member of a steel bridge.

From the tabulated results of experiment such as are shown in Fig. 128, E is found best, not by using the figures at any one point on the graph, but by taking the differences in stretch and load between two points, such as B and C, on the straight line drawn through several points obtained as a result of experiment. Thus errors which arise particularly at low values of load are eliminated. Thus increase of load from B to C=55 lb.—10 lb. Increase of stretch B to C is 0·069 in.—0·010 in.=0·059 in.

$$E=\frac{\text{increase in stress intensity}}{\text{increase in stretch } \textit{per inch}}=\frac{45 \text{ lb.}\div 0{\cdot}00237 \text{ sq. in.}}{0{\cdot}059 \text{ in.}\div 90{\cdot}7 \text{ in.}}$$

$$=29{,}190{,}000 \text{ lb. per sq. in.}$$

Similarly, for elastic compressive stress and strain, the relation (6) would hold, but the compressive strain would be (*decrease* in length)/(original length). But in the case of compressive stress we cannot by experiment get large compressive strains on long slender pieces or wires, as buckling occurs.

The corresponding constant for shear stress and strain is called the *modulus of rigidity*, for which the standard symbol is N (or alternatively C), so that,

$$N=\text{shear stress intensity } (q)/\text{shear strain } (\phi),$$

the shear strain ϕ being the angular distortion measured in radians (see Fig. 126).

98. Typical Stress-Strain Graph. Fig. 129 shows a typical graph for a tensile test of a piece of mild steel, an

important ductile material. The loads have been reduced to stress intensities by dividing them by the cross-sectional area of the test-piece, and the extensions have been reduced to proportional or fractional tensile strains by dividing them by the lengths over which they occurred. The graph differs from a load-extension graph, such as Fig. 127, only in the matter of scale. (The elastic stage is rather difficult to illustrate well on such a graph without exaggerating the very small strains so as to suggest a value of E

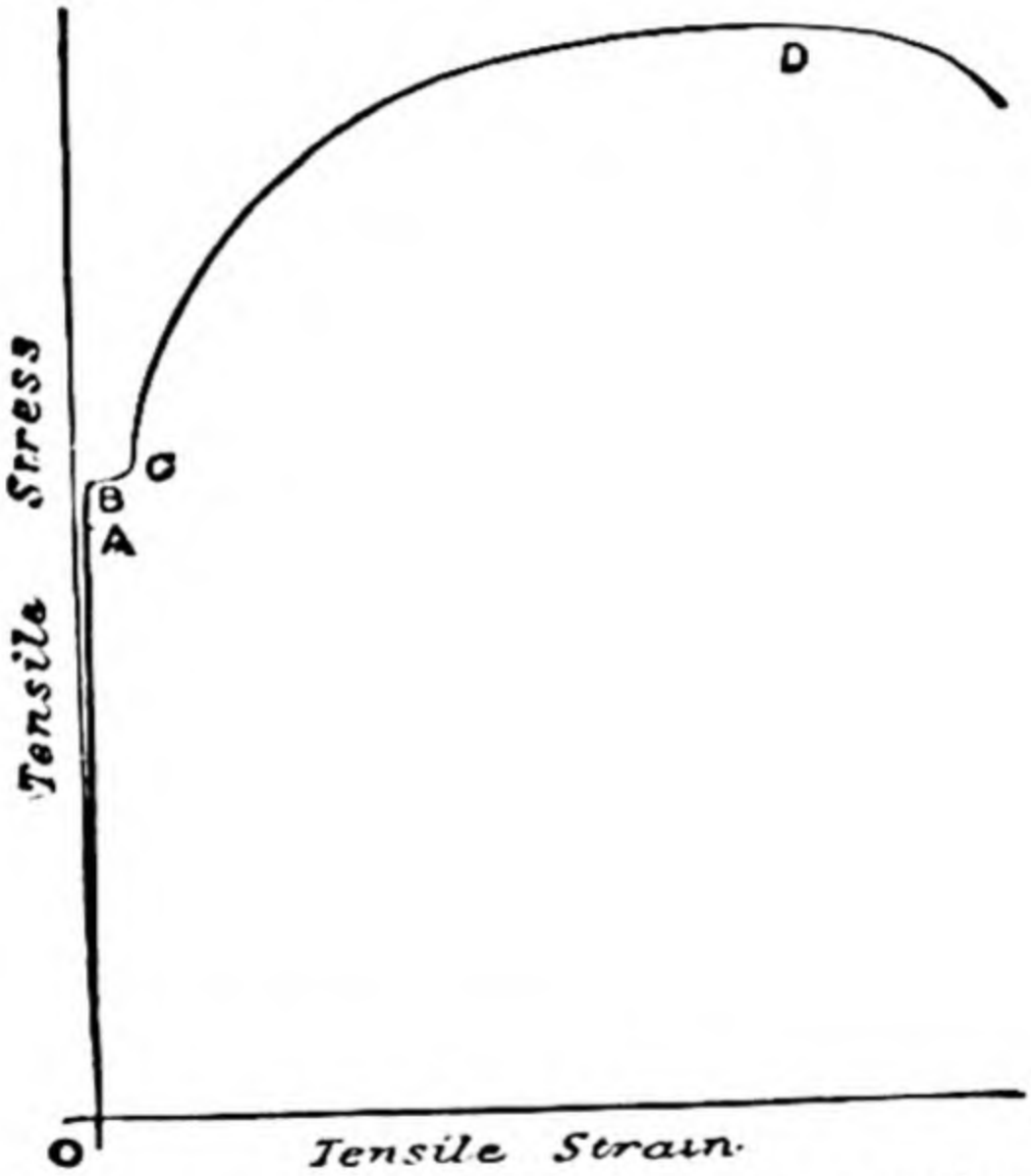


FIG. 129.—Stress-strain graph for ductile steel.

which is much too low, but Fig. 129 is about correct to scale and so gives a correct impression.) The point A marks approximately the *limit of proportionality* of stress to strain, and generally is practically at the *limit of elasticity* which marks the intensity of stress up to which the material is elastic. For stresses up to the elastic limit, the material returns to its former length on removal of the load, and beyond the limit, *permanent set* begins. The point B shows the "yield point" at which plastic yielding begins and

continues for a time to the point C without further increase in the load. This yielding is not found in all ductile materials, but is a marked feature in the softer irons and steels. The point D shows the *ultimate* or *maximum* stress intensity or *ultimate tensile strength* or tenacity reckoned on the original area and not on the reduced area of section which a test-piece has after plastic extension.

Non-ductile *brittle* or harder materials differ from mild steel in their behaviour under tensile stress in having either less or no plastic extension stage and no yield point. Ordinary cast iron and hard steel are examples of this.

99. Ultimate Strength and Factor of Safety. Ultimate tensile strength has already been referred to as the breaking tensile load (in a test) divided by the area of cross-section of the specimen. It is obvious that no such intensity of stress must be allowed to fall on a part of a machine or a structure. For it must not break, nor must it be strained by the large amounts by which ductile materials are stretched in a test. The allowable or *working stress* should, in fact, always be far below the ultimate strength and also almost always below the stress corresponding to the elastic limit. The number of times that the ultimate strength is greater than the working stress intensity is called the *Factor of Safety*, or

$$\text{Factor of Safety} = \frac{\text{ultimate strength}}{\text{working stress intensity}}$$

So that

$$\text{Allowable working stress intensity} = \frac{\text{ultimate strength}}{\text{factor of safety}}$$

The factor of safety to be adopted depends upon many circumstances and it often covers many contingencies very imperfectly known.

In respect of metals which strain plastically under high intensities of stress, it is not possible to state ultimate strengths under compressive stress; but in the case of brittle metals, particularly cast iron, which is largely used to resist pressure, the compressive ultimate strength is important.

The following table gives some idea of the ultimate

strengths of several of the metals in common use, but materials having a particular name vary widely in their strengths and often more precise information is required of the actual materials to be employed. This is particularly true in regard to cast irons and steels of widely differing kinds.

Material	Ultimate tensile strength in tons per sq. in.	Young's modulus in tons per sq. in.
Wrought iron	20 to 24	13,000
Mild steel (structural)	28 to 32	13,500
Steel wire	70 to 100	13,500
Cast iron	7 to 10	8,000
Copper (hard drawn)	20	6,500
Brass	8	5,500
Aluminium (rolled)	10	4,400

Crushing strength of cast iron, 40–50 tons per sq. in.
Modulus of rigidity of steel, 5500 to 6500 tons per sq. in.

100. Stress in Thin Shells. A thin circular cylindrical shell containing fluid at a pressure higher than that outside is subject to tension. The intensity of this stress in two important directions is easily found.

Hoop or Circumferential Tension. Let Fig. 130 represent

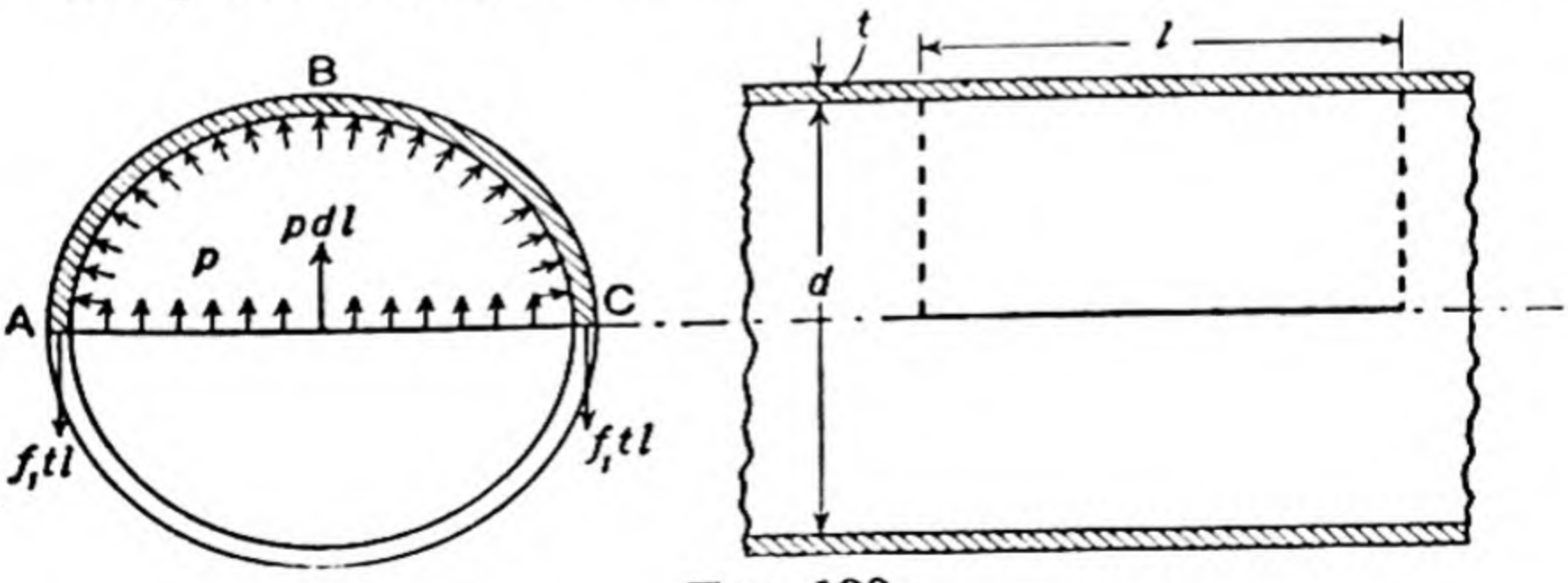


FIG. 130.

a shell of diameter d and thickness t and let p be the intensity of the internal pressure. Considering an axial length l, the shell has exerted upon it forces tending to

pull it asunder into two half-cylinders. In the half ABC across a diametrical plane AC, the total force exerted by the fluid pressure is

$$P = p \times d \times l \quad . \quad . \quad . \quad . \quad (1)$$

which will be in pounds if p is in pounds per square inch, d and l in inches. The total force resisting such bursting, offered by the material, will act tangentially through A and C distributed along the length l; and if the intensity of this tension is f_1 the total resisting force will be

$$f_1 \times \text{area of section of metal} = f_1 \times 2t \times l \quad . \quad (2)$$

there being two strips of material each of thickness t and length l. The total bursting force and total resistance being equal

$$pdl = 2f_1tl \quad . \quad . \quad . \quad . \quad (3)$$

$$f_1 = pd/2t \text{ or } p = 2f_1t/d \quad . \quad . \quad . \quad . \quad (4)$$

(It may be noted that if a cylindrical shell is rotating about its axis as in Art. 73, the radial force exerted by each piece of shell of unit area and thickness t on the adjoining shell is $p = wt\omega^2 r/g$, and substituting this in (4), $f_1 = \omega\omega^2 r^2/g$ or $w\omega^2 v^2/g$ as in (5) of Art. 73, where the symbols r, v, w, and ω have the same meanings as in Art. 73).

Longitudinal Tension. Across any section AB, perpendicular to the axis of the cylinder, Fig. 131, the fluid

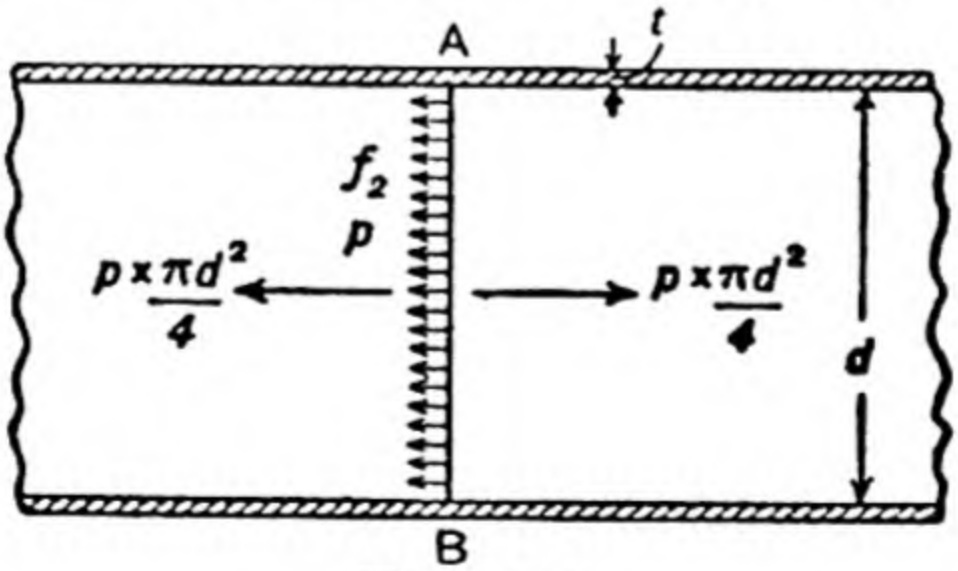


FIG. 131.

pressure tending to burst the cylinder asunder into two cylindrical parts, is

$$p \times \pi d^2/4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The total resisting forces will be equal to the intensity of longitudinal tension, f_2 say, multiplied by the area of cross-section perpendicular to the axis, which is $\pi d \times t$, so that the total force is

$$\pi dt \times f_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

And equating these balanced forces,

$$p \times \pi d^2/4 = f_2 \times \pi dt \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or

$$f_2 = pd/4t \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which is half the value of f_1, the hoop or circumferential stress intensity (see (4)).

If p is in lb. per sq. in. and t is inches, then d must also be in inches.

EXAMPLE 1. A cylindrical seamless shell 7 feet internal diameter has to stand an internal pressure of 200 lb. per square inch, the plates being $\frac{7}{8}$ inch thick. Find the intensity of hoop stress.

$$f_1 = \frac{pd}{2t} = \frac{200 \times 7 \times 12}{2 \times 0{\cdot}875} = 9600 \text{ lb. per square inch.}$$

EXAMPLE 2. What thickness of cast-iron pipe 10 inches internal diameter will be required to stand an internal pressure of 50 lb. per square inch if the stress in the pipe is not to exceed 1000 lb. per square inch?

$$pd = 2ft$$

$$\therefore 50 \times 10 = 2 \times 1000 \times t$$

$$t = \tfrac{500}{2000} = \tfrac{1}{4} \text{ inch.}$$

EXAMPLES X

1. A round bar of steel 1 inch in diameter is subjected to a total pull of 10 tons. Find the total stretch on a length of 8 inches and the fractional strain if E = 13,120 tons per sq. in.

2. A steel wire $\frac{1}{8}$ inch diameter and 10 feet long stretches 0·83 inch under a pull of 250 pounds. Find Young's modulus of elasticity for the material.

3. The following results were obtained from a tension test of a copper wire 0·05 inch diameter and 90 inches long :

Load (lb.)	0	5	10	15	20	25	30	34	36·5	39	41·5	44	46·5	50	55
Stretch (in.)	0	0·015	0·03	0·045	0·06	0·14	0·34	1·12	1·80	2·69	3·64	8·64	6·53	8·04	10·21

Plot the load-extension diagram and find the value of Young's Modulus of Elasticity for the copper wire.

4. What load hung on a vertical wire 15 feet long and 0·1 inch diameter will cause it to stretch 0·125 inch if E = 30,000,000 lb. per sq. in. ?

5. A steel tie-rod 40 feet long and 1 inch diameter is subjected to a pull of 8 tons. Calculate (1) the total elongation if Young's Modulus is 13,000 tons per sq. in., and (2) the factor of safety if the ultimate tensile strength is 30 tons per sq. in.

6. A hollow cast-iron strut is 10 inches external and 8 inches internal diameter and 10 feet long. How much will it shorten under a load of 60 tons ? Take E as 8000 tons per square inch.

7. The overhead line of a trolley-bus system is of hard-drawn copper wire $\frac{3}{8}$ inch diameter, having an ultimate strength of 20 tons per sq. in. Allowing a factor of safety of 6, calculate the allowable pull on the wire and find the corresponding elongation on a span of 120 feet if E = 6500 tons per sq. in.

8. A cylindrical steam boiler is 6 feet internal diameter, and is made up of plates $\frac{5}{8}$ inch thick. If the internal steam pressure is 150 lb. per square inch, what is the intensity of hoop stress in the plates ?

9. What thickness of cast-iron pipe 16 inches internal diameter will be required to stand an internal pressure of 80 lb. per square inch if the intensity of stress in the material is not to exceed 1000 lb. per square inch ?

CHAPTER XI

STRUCTURES

101. Framed Structures. A framed structure, or simply a *frame*, is a structure consisting of a number of bars fastened together by what are in effect, though often not in fact, hinged joints. The separate bars are members of the frame and are long relatively to their lateral dimensions. Loads are applied to such structures mainly at their joints, the members being subjected only to nearly axial forces of pull or thrust. The centre lines by which the frames are diagrammatically represented are the lines of action of the forces acting at the joints. The joints are often not hinged but are riveted, and frequently two or more members separated by joints form one continuous piece of metal. Nevertheless, the stresses in the members are calculated as if all the bars were separate pieces freely hinged together at the joints. The justification of this depends upon the circumstances, and the estimation of the errors involved in such assumptions is a matter for more advanced study of this subject.

We shall consider only frames which have just a sufficient number of members to prevent deformation or collapse under the loads. Such frames are called perfect frames. With fewer members, a frame may develop into a mechanism (such as in Fig. 66 or Fig. 67); and with more members, one or more are redundant and the forces in the several members are dependent on their lateral dimensions, and the problem of finding the stresses in such frames is beyond the scope of this book. We shall further limit our consideration to frames in which the centre lines of the members and loads all lie approximately in one plane.

102. Determination of Stresses in Members of Perfect Frames. The stresses in individual members of a perfect frame, all members being either struts or ties, are determined by application of the principles of statics stated and used in Chapter II. Either graphical or arithmetical methods or a combination of both may be used, but in any case the

following are the guiding principles. (1) The frame as a whole is a rigid body, and the external forces acting upon it (loads and supporting forces) form by themselves a system of forces (generally non-concurrent) in equilibrium. (2) The pulls and thrusts which the several members meeting at a joint exert on, say, the pin of the joint form a system of concurrent forces in equilibrium. (3) As pointed out in Art. 28 (method of sections), any portion of the structure may be taken as a rigid body held in equilibrium by external forces acting upon it together with the forces exerted upon it, through members, by the remainder of the structure.

In applying the principles of equilibrium to the whole structure, to a joint, to a single member, or to any portion of the structure, it is desirable to keep in mind that we are considering the equilibrium of that portion or of the forces exerted upon it. Then we do not include any force which it may exert on other parts. Some simple examples of determination of stresses in framed structures have already been given in Art. 28, and we now proceed to amplify the graphical methods.

103. Stress Diagrams. When all the external forces, viz. loads and supporting forces, acting on a frame have been determined, we can proceed to draw the vector force polygons for each of the hinges, the pins of which are each in equilibrium under a number of balanced concurrent forces, viz. those exerted on the joint by the members meeting at it. The vector force polygons could be fitted together to form a single vector diagram for the whole of the frame. Each line in such a diagram would represent the *two* equal and opposite forces exerted by a member on the two joints at its opposite ends, and would be a component side of two of the separate force polygons for the joints. An example will make this clear.

Fig. 132 represents at (*a*) a simple frame consisting of five bars, the joints of which have been denoted at (*a*) by 1, 2, 3, and 4. The frame stands in the vertical plane, and carries a known vertical load, W, at the joint 3; it rests on supports on the same level at 1 and 4. The force W is denoted in Bow's notation by the letters PQ. The reactions at 1 and 4, named RP and QR respectively, have been

found by a funicular polygon corresponding to the vector diagram at (*b*), as described in Art. 34 and illustrated in Fig. 20.

Letters S and T have been used for the two remaining spaces. When the upward vertical force RP at the joint 1 is known, the triangle of forces *rps* at (*c*) can be drawn by making *rp* proportional to RP as in (*b*), and completing the

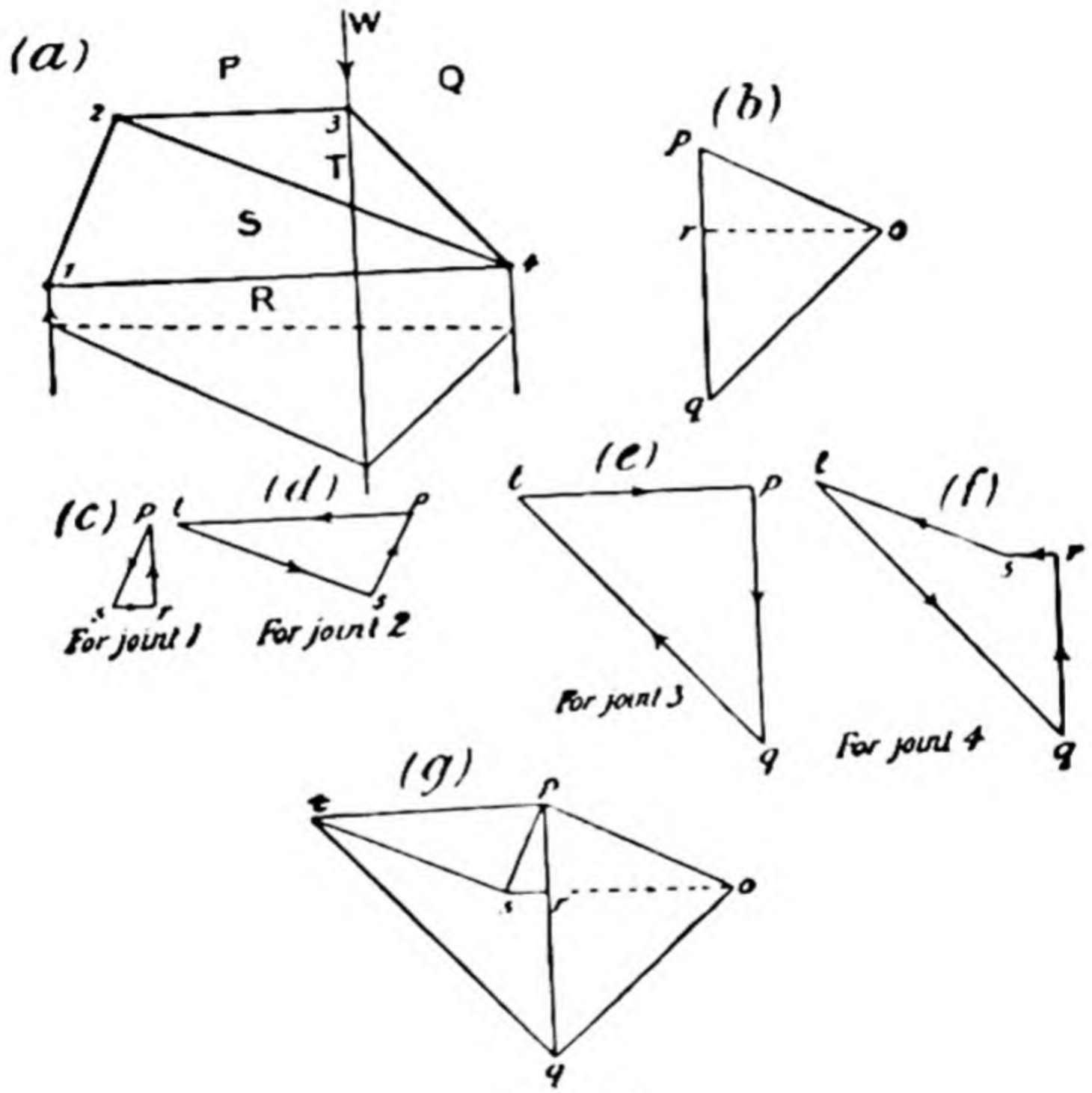


FIG. 132.

triangle by sides parallel to PS and SR (*i.e.* to the bars 12 and 14) respectively. After this triangle has been drawn, one of the three forces acting at the joint 2 is known, viz. SP acting in the bar 12, being equal and opposite to PS in (*c*). Hence the triangle of forces *spt* at (*d*) for the joint 2 can be drawn. Next the triangle *tpq* at (*e*) for joint 3

can be drawn, *tp* and *pq* being known; the line joining *qt* will be found parallel to the bar QT if the previous drawing has been correct; this is a check on the accuracy of the results. Finally, the polygon *qrst* at (*f*) for joint 4 may be drawn, for all four sides are known in magnitude and direction from the previous polygons. The fact that when drawn to their previously found lengths and directions they form a closed polygon constitutes a check to the correct setting out of the force polygons. The arrow-heads on the sides of the polygons denote the directions of the forces on the particular joint to which the polygon refers.

It is to be noticed in Fig. 132 that in the polygons (*b*), (*c*), (*d*), (*e*), and (*f*), drawn for the external forces on the frame and the forces at the various joints, each side, whether representing an external or internal force, has a line of equal length and the same inclination in some other polygon.

For example, *sr* in (*c*) corresponds to *rs* in (*f*), and *pt* in (*d*) with *tp* in (*e*). The drawing of entirely separate polygons for the forces at each joint is unnecessary; they may all be included in a single figure, such as (*g*), which may be regarded as the previous five polygons superposed, with corresponding sides coinciding. Such a figure is called a stress diagram for the given frame under the given system of external loading. It contains (1) a closed vector polygon for the system of external forces in the frame, (2) closed vector polygons for the (concurrent) forces at each joint of the structure.

As each vector representing the internal force in a member of the frame represents two equal and opposite forces, arrow-heads on the vectors are useless or misleading, and are omitted.

Distinction between Tension and Compression Members of a Frame. A member which is in tension is called a "tie," and is subjected by the joints at its ends to a pull tending to lengthen it. The forces which the member exerts on the joints at its ends are equal and opposite *pulls*, tending to bring the joints closer together.

A member which is in compression is called a "strut"; it has exerted upon it by the joints at its ends two equal and

opposite pushes or thrusts tending to shorten it. The member exerts on the joints at its ends equal and opposite "outward" thrusts tending to force the joints apart.

The question whether a particular member is a "tie" or a "strut" may be decided by finding whether it pulls or thrusts at a joint at either end. This is easily discovered if the direction of any of the forces at that joint is known, since the vector polygon is a closed figure with the last side terminating at the point from which the first was started. *E.g.* to find the kind of stress in the bar 24, or ST (Fig. 132). At joint 4, QR is an upward force; hence the forces in the polygon $qrst$ must act in the directions $\overrightarrow{qr}$, $\overrightarrow{rs}$, $\overrightarrow{st}$, and $\overrightarrow{tq}$; hence the force ST in bar 24 acts at joint 4 in the direction *s* to *t*, *i.e.* the bar pulls at joint 4, or the force in ST is a tension. Similarly, the force in bar 23, or PT, acts at joint 3 in a direction *tp*, *i.e.* it pushes at joint 3, or the force in bar 23 is a compressive one.

Another Method. Knowing the direction of the force *rp* at joint 1 (Fig. 132), we know that the forces at joint 1 act in the directions *rp*, *ps*, and *sr*, or the vertices of the vector polygon *rps* lie in the order $r-p-s$.

The corresponding lines RP, PS, and SR in the space diagram are in *clockwise* order round the point 1. This order, clockwise or counter-clockwise (but in this instance clockwise), is the same for every joint in the frame. If it is clockwise for joint 1, it is also clockwise for joint 2. Then the vertices of the vector polygon for joint 2 are to be taken in the cyclic order $s-p-t$, since the lines SP, PT, and TS lie in clockwise order round the joint 2, *e.g.* the force in bar 23, or PT, is in the direction $\overrightarrow{pt}$, *i.e.* it *thrusts* at joint 2.

This characteristic order of space letters round the joints is a very convenient method of picking out the kind of stress in one member of a complicated frame. Note that it is the characteristic order of space letters round a joint that is constant—*not* the direction of vectors round the various polygons constituting the stress diagram. For a counter-clockwise order the stress diagram would differ in shape from that in Fig. 132 (*g*). We return to this characteristic order in Art. 105.

Reciprocal Figures. The frame or space diagram of, say, Fig. 132, and the stress or vector diagram, form

reciprocal figures which have certain reciprocal properties; to each node or vertex from which lines radiate in one figure there is a corresponding closed polygon in the other, bounded by sides corresponding to the radiating lines and respectively parallel to them. To each line joining two nodes in either figure there is a corresponding line in the other forming a side to the polygons corresponding to the two nodes.

104. Warren Girder. A second example of a simple stress diagram is shown in Fig. 133, viz. that of a common type of frame called the Warren girder, consisting of a number of bars jointed together as shown, all members

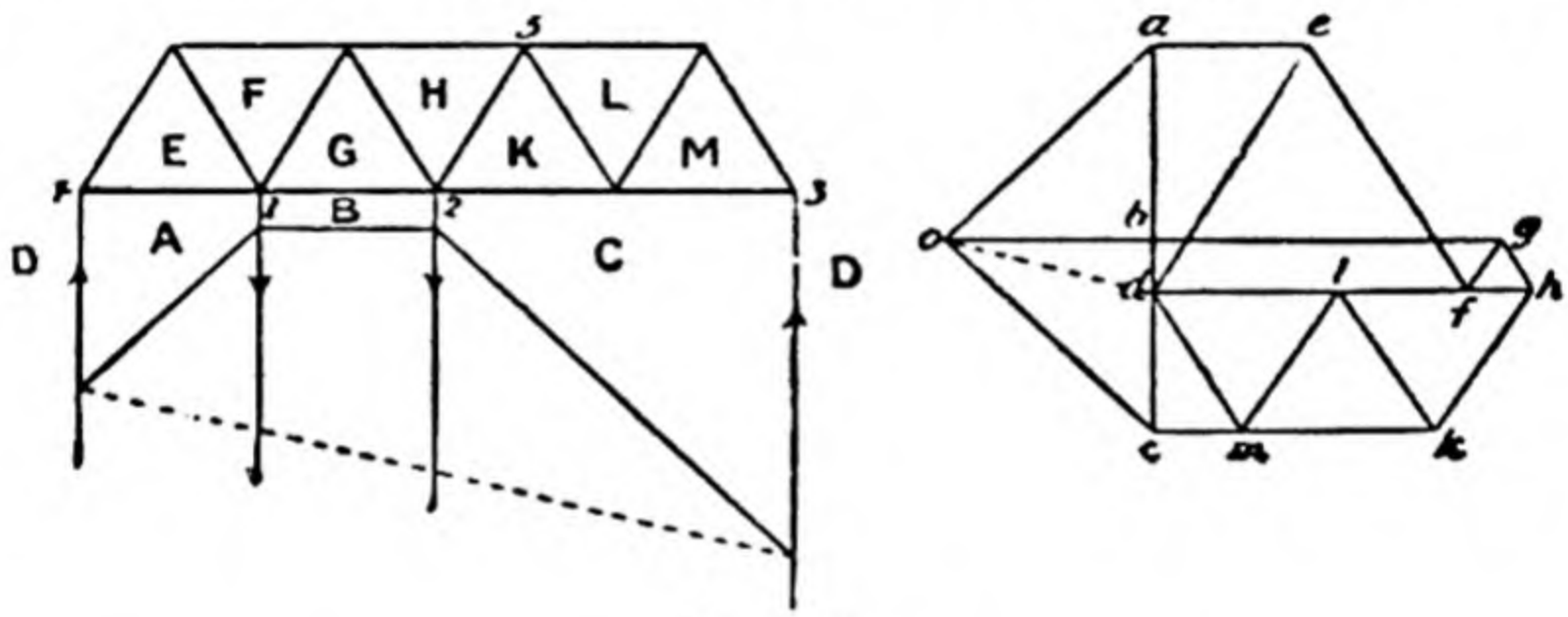


FIG. 133.

generally being of the same lengths, some horizontal, and others inclined 60° to the horizontal.

Two equal loads, AB and BC, have been supposed to act at the joints 1 and 2, and the frame is supported by vertical reactions at 3 and 4, which are found by a funicular polygon. The remaining forces in the bars are found by completing the stress diagram *abc . . . klm*.

Note that the force AB at joint 1 is downward, *i.e.* in the direction *ab* in the vector diagram corresponding to a counter-clockwise order, A to B, round joint 1. This is, then, the characteristic order (counter-clockwise) for all the joints, *e.g.* to find the nature of the stress in KL, the order of letters for joint 5 is K to L (counter-clockwise), and referring to the vector diagram, the direction *k* to *l* represents a thrust of the bar KL on joint 5; the bar KL is therefore in compression.

105. Simple Roof-frame. Fig. 134 shows a simple roof-frame and its stress diagram when carrying three equal vertical loads on three joints and supported at the extremities of the span.

The reactions DE and EA at the supports are each obviously equal to half the total load, *i.e.* *e* falls midway

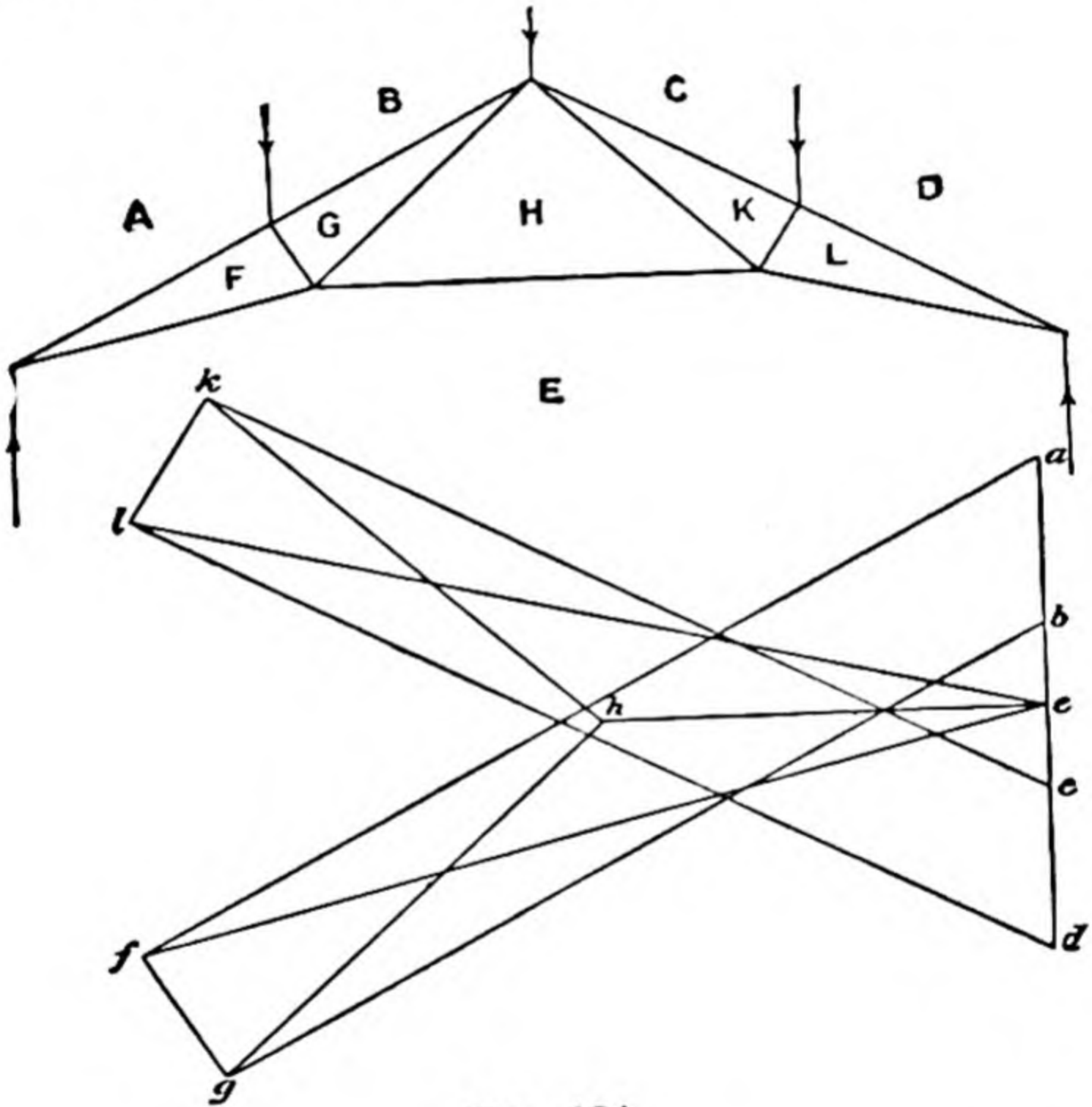


FIG. 134.

between *a* and *d* in the stress diagram. The correct characteristic order of the letters round the joints (Art. 103) is, with the lettering here adopted, clockwise.

To distinguish between Ties and Struts from the Stress Diagram. Knowing the direction of, say, the force EA (upward) at the joint EAF, it is evident from Fig. 134 that the correct order of letters in the vector polygon for this

joint is *eaf* (not *efa*), hence the force at this joint exerted by the rafter AF is represented by *af* (not *fa*), and is a thrust, *i.e.* the member is a strut. The correct order of sides *eaf* being *ea*, *af*, *fe* the corresponding order of the lines EA, AF, FE radiating from this joint is a *clockwise* order. When this order is clockwise for one joint it immediately follows that it must be the same for the neighbouring joints, for a thrust, *af*, must be associated with a balancing thrust, *fa*, at the next joint of the rafter. Similarly, it follows that the correct order is clockwise for *all* the joints. Hence if we wish to know whether the member HK, say, is a strut or a tie, we know that for the joint HKLE, the force in HK is in the direction *hk* (not *kh*), and reference to the vector diagram shows that the direction *hk* is a *pull* at the joint HKLE, *i.e.* HK is a *tie*.

Fig. 135 represents the stress diagram for exactly the same frame diagram and lettering as Fig. 134, but is the

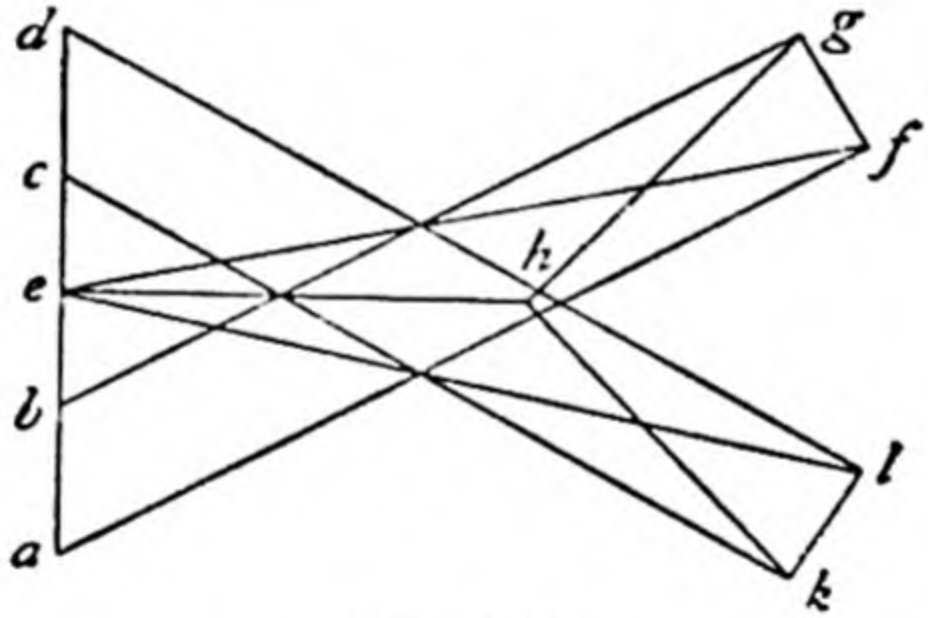


Fig. 135.

counter-clockwise vector diagram, *e.g.* the left-hand reaction AE is now represented by *ae* (instead of *ea*), and the force of KH at the joint HELK is represented by *kh* (instead of *hk*), which still indicates the member to be a tie.

106. Wind Loads. In Figs. 132, 133, and 134, the loads and reactions or supporting forces are vertical. If the external forces of loads and supports are not vertical, the procedure of drawing the stress diagram is not in any way altered, but for inclined loads the supporting forces will not generally both be vertical. By rollers or a smooth sliding

surface which offers no appreciable resistance to horizontal movement in the plane of the frame, a practically vertical reaction or supporting force can be ensured at one support, and in that case the other supporting force must have a horizontal component which just equals and is opposite to the total horizontal component of the loads. To find the supporting forces we have a choice of three methods.

(1) This method has been illustrated in Example 1 of Art. 27. By equating opposite moments about the fixed or hinged support of the roof, we eliminate any moment of the inclined reaction there and so find by one simple equation the other reaction (at the free end), which is vertical. For its moment about the hinge must be equal and opposite to the moment of all the loads about that axis. When the vertical supporting force is known, the force polygon for the external forces on the frame can be drawn, for only the closing side remains unknown, and this, when drawn in, gives the (inclined) supporting force at the hinged end.

(2) The same result can be obtained by purely graphical construction by using the funicular polygon for two unknown equilibrants as explained in Art. 33. This is illustrated in Fig. 136 for a roof-frame, with vertical load combined with wind loads perpendicular to the main rafter on one side, viz. that of the fixed or hinged support. In order to keep the vector force diagram clearer, the vertical and inclined loads have been combined (by parallelograms) at the joints on the rafter on the left side of the roof. After the vector polygon *abcdefgh* has been set off to scale, a pole *o* is chosen and joined to the angles at *b*, *c*, *d*, *e*, *f*, and *g*. Then the funicular polygon is drawn, *starting from the hinge* at the fixed side, viz. (B)O is drawn parallel to *bo*, (C)O to *co*, (D)O to *do*, and so on, until finally (G)O parallel to *go* meets a vertical line through the free end. (This intersection in Fig. 136 happens to fall on the free end; but for a different choice of the pole *o*, it would generally fall above or below it.) This intersection is then joined by the line TO to the hinge, and a parallel line through *o* to intersect the vertical line through *h* gives *ht* as the magnitude of the supporting force HT, and joining *t* to *a* gives, by the vector *ta*, the magnitude and direction of the force TA at the hinge. The stress diagram is then

drawn and needs no comment except that *gh* does not enter into it. It merely increases *ht*, which would otherwise be *gt*. Similarly, *ab* only slightly affects the stresses since *ab* mainly adds to *ta*, but the two lines are not *quite* in the same direction.

(3) The supporting forces might also be found by first

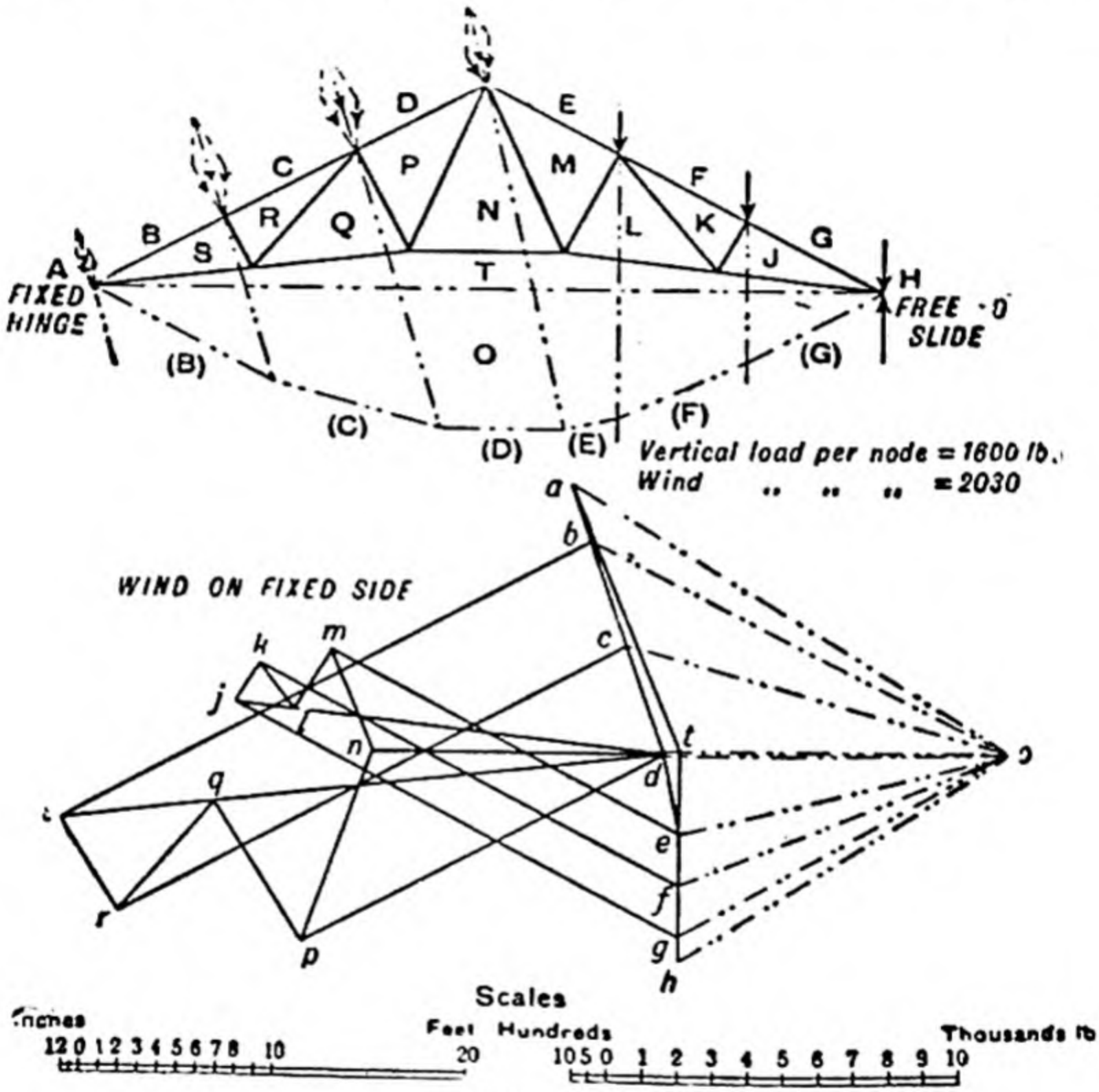


FIG. 136.

finding the resultant of all the loads in magnitude, direction, and position by combination in successive pairs or by the funicular polygon (*i.e.* by producing (B)O and (G)O to meet and drawing through their intersection a line parallel to *ah*). The external forces on the frame would thus be reduced to three, and hence they would be concurrent

(Art. 29). By producing the line of action of the resultant load to meet the vertical line through the free end, we should get the point of concurrency. A line from it to the hinge gives the direction and position of the inclined supporting force, and the vector force polygon could then be completed since the direction of the two supporting forces would be known, viz. *at* is parallel to the inclined one and *ht* is vertical.

But this method is scarcely to be recommended in such a case because the intersection to give the point of concurrency is one of two lines at a small acute angle. If the wind loads only were considered, this method would be less open to objection. Sometimes, separate stress diagrams are drawn, one for wind loads and another for the vertical loads, and then their effects are added for each member.

107. Loaded Chains and Cords. Although suspended cables, cords, or chains do not come within the general meaning of the word frame, stress diagrams can be drawn for a structure consisting partly of perfectly flexible chains or ropes, provided the loads are such as will cause only tension in flexible members.

Consider a flexible cord or chain X123Y (Fig. 137), suspended from points X and Y, and having vertical loads of W_1, W_2, and W_3 suspended from points 1, 2, and 3 respectively. Denoting the spaces according to Bow's notation by the letters A, B, C, D, and O, as shown above, the tensions in the strings X1 or AO and 1 2 or BO must have a resultant at 1 equal to W_1 vertically upward, to balance the load at 1. If triangles of forces, *abo*, *bco*, and *cdo*, be drawn for the points 1, 2, and 3 respectively, the sides *bo* and *co* appear in two of them, and, as in Art. 103, the three vector triangles may be included in a single vector diagram, as shown at the right-hand by the figure *abcdo*.

The lines *ao*, *bo*, *co*, and *do* represent the tensions in the string crossing the spaces A, B, C, and D respectively. If a horizontal line, *o*H, be drawn from *o* to meet the line *abcd* in H, the length of this line represents the horizontal component of the tensions in the strings, which is evidently constant throughout the whole. (The tension changes only from one space to the neighbouring one by the vector addition of the intermediate vertical load.) The pull on

the support X is represented by *ao*, the vertical component of which is *a*H; the pull on Y is represented by *od*, the vertical component of which is H*d*.

A comparison with Art. 34 will show that the various sections of the string X123Y are in the same lines as the sides of a funicular polygon for the vertical forces W_1, W_2, and W_3, corresponding to the pole *o*. If different lengths of string are attached to X and Y and carry the same loads, W_1, W_2, and W_3, in the lines AB, BC, and CD respectively, they will have different configurations; the longer the string, the steeper will be its various slopes, corresponding

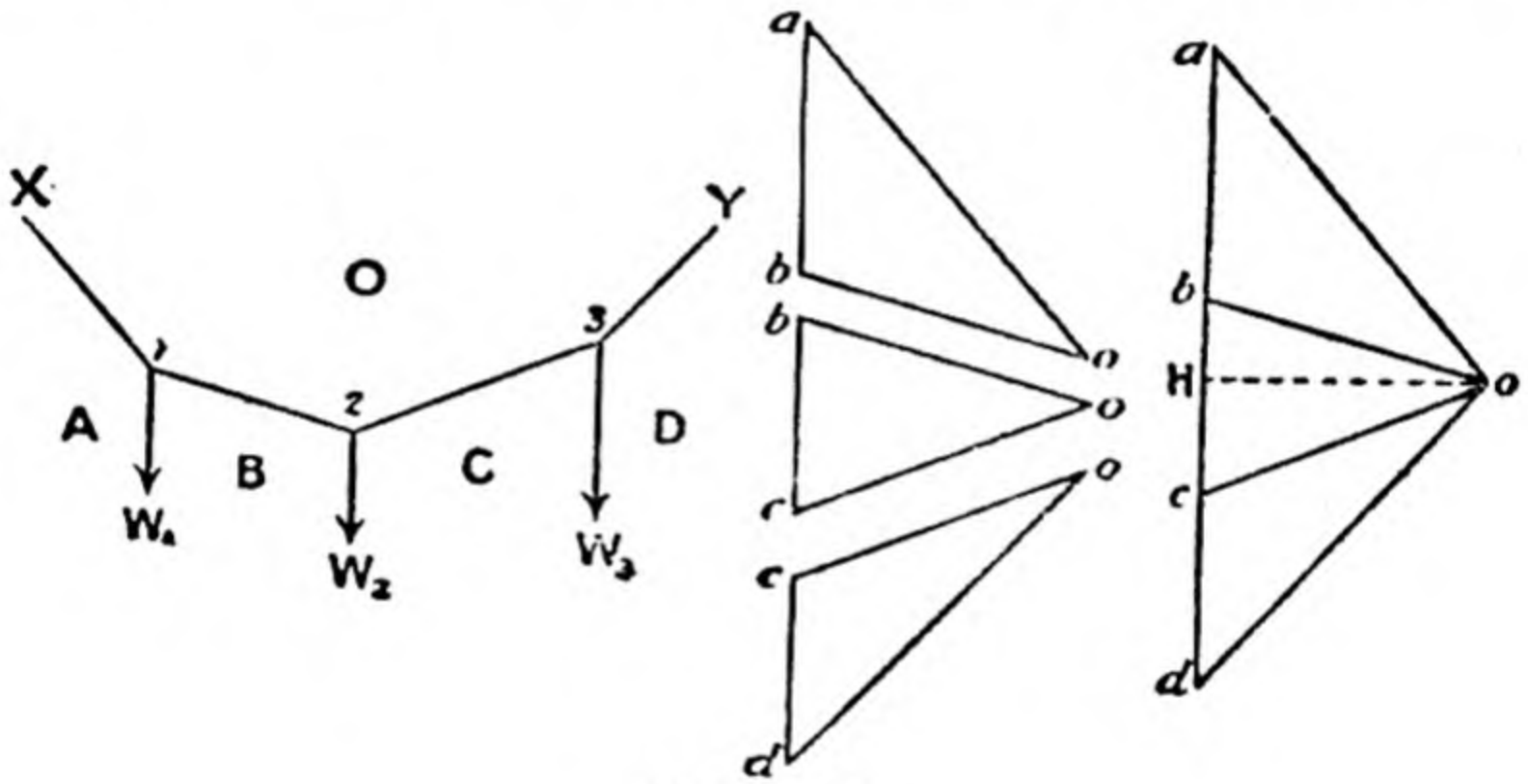


FIG. 137.

to shorter pole distances, H*o*, *i.e.* to smaller horizontal tensions throughout. A short string will involve a great distance of the pole *o* from the line *abcd*, *i.e.* a great horizontal tension, with smaller inclinations of the various sections of the string. The reader should sketch for himself the shape of a string connecting X to Y, with various values of the horizontal tension H*o*, the vertical loads remaining unaltered, in order to appreciate fully how the tensions increase in a shortened string.

A chain with hinged links, carrying vertical loads at the joints, will occupy the same shape as a string of the same length carrying the same loads. Such chains are used in suspension bridges.

The shape of the string or chain to carry given loads in

assigned vertical lines of action can readily be found for any given horizontal tension, H*o*, by drawing the various sections parallel to the corresponding lines radiating from *o*, *e.g.* AO or X1 parallel to *ao* (Fig. 137).

EXAMPLE 1. A string hangs from two points, X and Y 5 feet apart, X being 3 feet above Y. Loads of 5, 3, and 4 lb. are attached to the string so that their lines of action are 1, 2, and 3 feet respectively from X. If the horizontal tension of the string is 6 lb., draw its shape.

The horizontal distance ZY (Fig. 138) of X from Y is—

$$\sqrt{5^2-3^2}=4 \text{ feet}$$

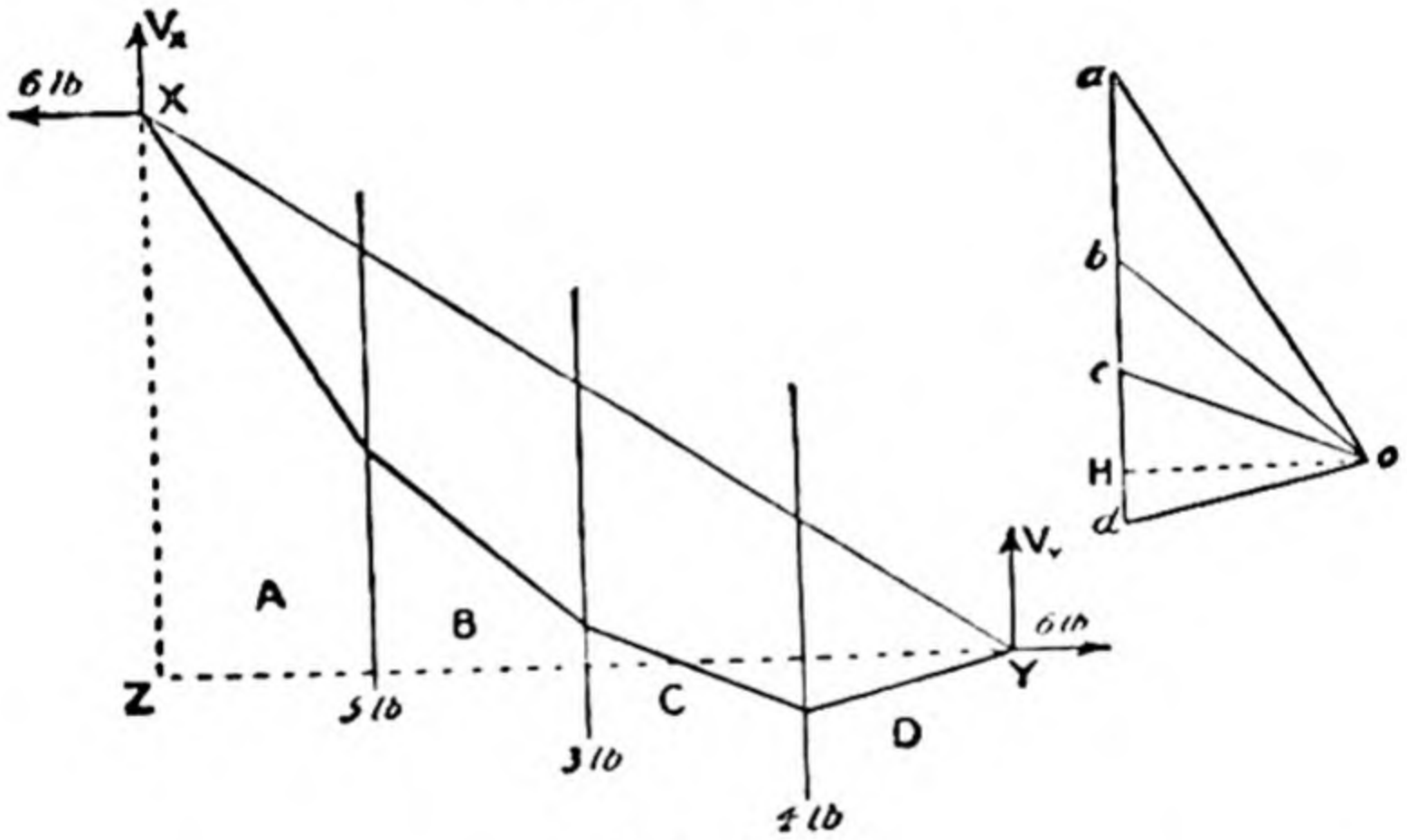


FIG. 138.

so that the three loads divide the horizontal span into four equal parts.

Let V_X and V_Y be the vertical components of the tension of the string at X and Y respectively.

The horizontal tension is constant, and equal to 6 lb. Taking moments about Y (Fig. 138)—

$$\underset{\text{Clockwise.}}{V_X \times 4 \text{ ft.}} = \underset{\text{Counter-clockwise.}}{(4\times1)+(3\times2)+(5\times3)+(6\times3)} \text{ lb.-feet.}$$

$$4V_X = 4+6+15+18 = 43 \text{ lb.-ft.}$$

$$V^X = \frac{43 \text{ lb.-ft.}}{4 \text{ ft.}} = 10{\cdot}75 \text{ lb.}$$

Since the vertical and horizontal components of the tension of the string at X are known, its direction is known. The direction of each section of string might similarly be found. Set out the vector polygon *abcd*, and draw the horizontal line H*o* to represent 6 lb. horizontal tension from H, *a*H being measured along *abcd* of such a length as to represent the vertical component 10·75 lb. of the string at X. Join *o* to *a*, *b*, *c*, and *d*. Starting from X or Y, draw in the lines across spaces A, B, C, and D, parallel respectively to *ao*, *bo*, *co*, and *do* (as in Art. 34). The funicular polygon so drawn is the shape of the string.

EXAMPLE 2. A chain is attached to two points, X and Y, X being 1 foot above Y and 7 feet horizontally from it. Weights of 20, 27, and 22 lb. are to be hung on the chain at horizontal distances of 2, 4, and 6 feet from X. The chain is to pass through a point P in the vertical plane of X and Y, 4 feet below, and 3 feet horizontally from X. Find the shape of the chain and the tensions at its ends.

Let V_X and V_Y be the vertical components of the tension at X and Y respectively, and let H be the constant horizontal tension throughout.

Taking moments about Y (Fig. 139)—

Clockwise. Counter-clockwise.

$$V_X \times 7 = (H \times 1) + (20 \times 5) + (27 \times 3) + (22 \times 1) \text{ lb.-ft.}$$

$$7V_X = H + 203 \text{ lb.-feet} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Taking moments about P of the forces on the chain between X and P—

Clockwise. Counter-clockwise.

$$V_X \times 3 = H \times 4 + (20 \times 1)$$

$$3V_X = 4H + 20 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{and } 28V_X = 4H + 812 \text{ from (1)}$$

$$\text{hence } 25V_X = 792 \text{ lb.-ft.}$$

$$V_X = 31{\cdot}68 \text{ lb.}$$

$$H = 7V_X - 203 = 221{\cdot}76 - 203 = 18{\cdot}76 \text{ lbs.}$$

Draw the open polygon of forces, *abcd* (a straight line), and set off *am* from *a* to the same scale, 31·68 lb. downwards.

From m, set off mo to represent 18·76 lb. horizontally to the right of m.

Then the vector $ao = am + mo$ = tension in the chain XZ, which pulls at X in the direction XZ. By drawing XZ parallel to ao, the direction of the first section of the chain is obtained, and by drawing from Z a line parallel to bo to meet the line of action BC, the second section is outlined. Similarly, by continuing the polygon by lines parallel to

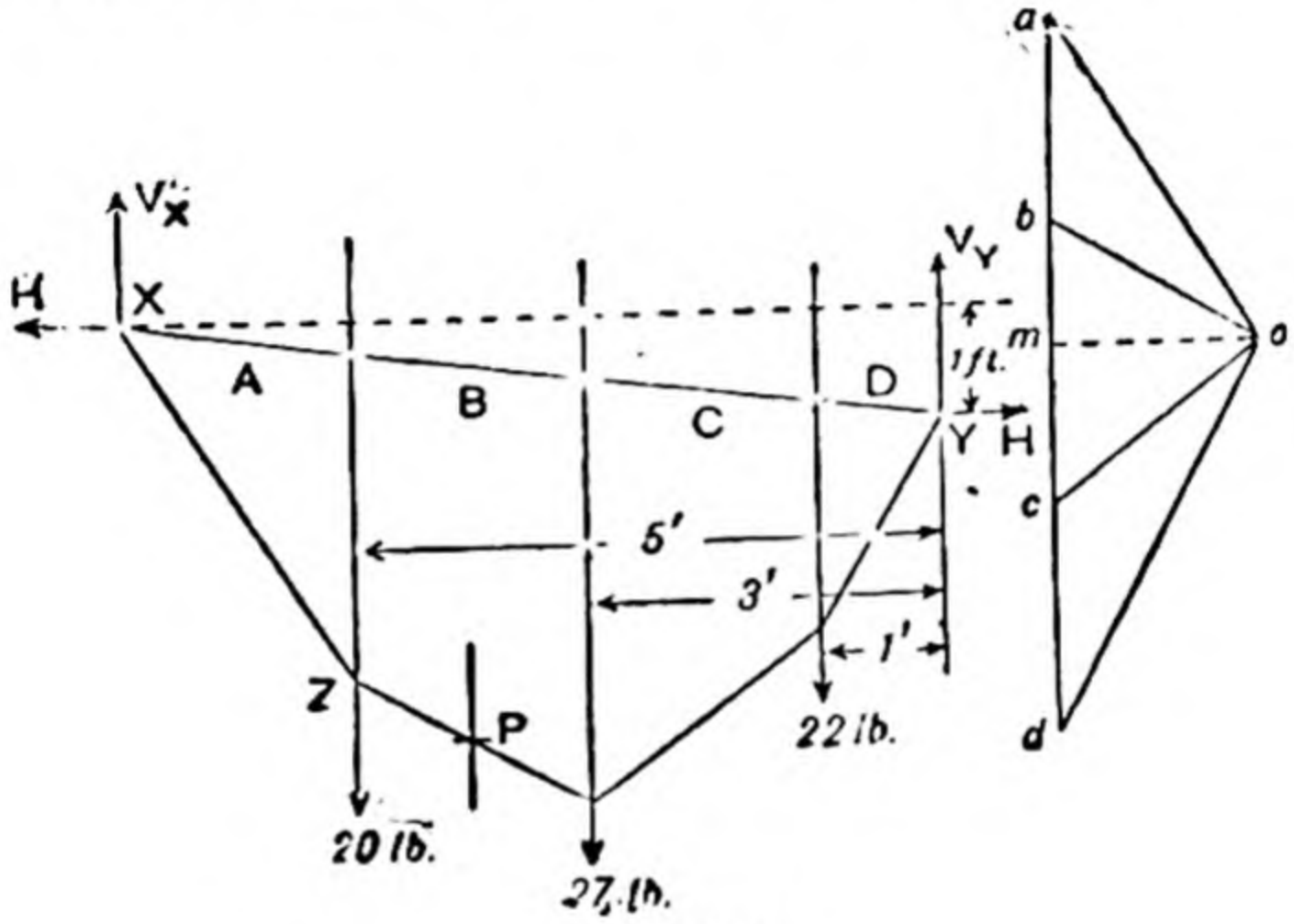


Fig. 139.

co and do, the complete shape of the chain between X and Y is obtained.

The tension ao at X scales 37 lb., and the tension od at Y scales 44 lb.

108. Distributed Load. If the number of points at which the same total load is attached to the string (Fig. 137) be increased, the funicular polygon corresponding to its shape will have a larger number of shorter sides, approximating, if the number of loads be increased indefinitely, to a smooth curve. This case corresponds to that of a heavy chain or cable hanging between two points with no vertical load but its own weight. If the dip of the chain from the

straight line joining the points of the attachment is small, the load per unit of horizontal span is nearly uniform provided the weight of chain per unit length is uniform. In this case, an approximation to the shape of the chain may be found by dividing the span into a number of sections of equal length and taking the load on each portion as concentrated at the mid-point of that section. The funicular polygon for such a system of loads will have one side more than the number into which the span has been divided; the approximation may be made closer by taking more parts. The true curve has all the sides of all such polygons as tangents, or is the curve inscribed in such a polygon.

The polygons obtained by treating a span as one and

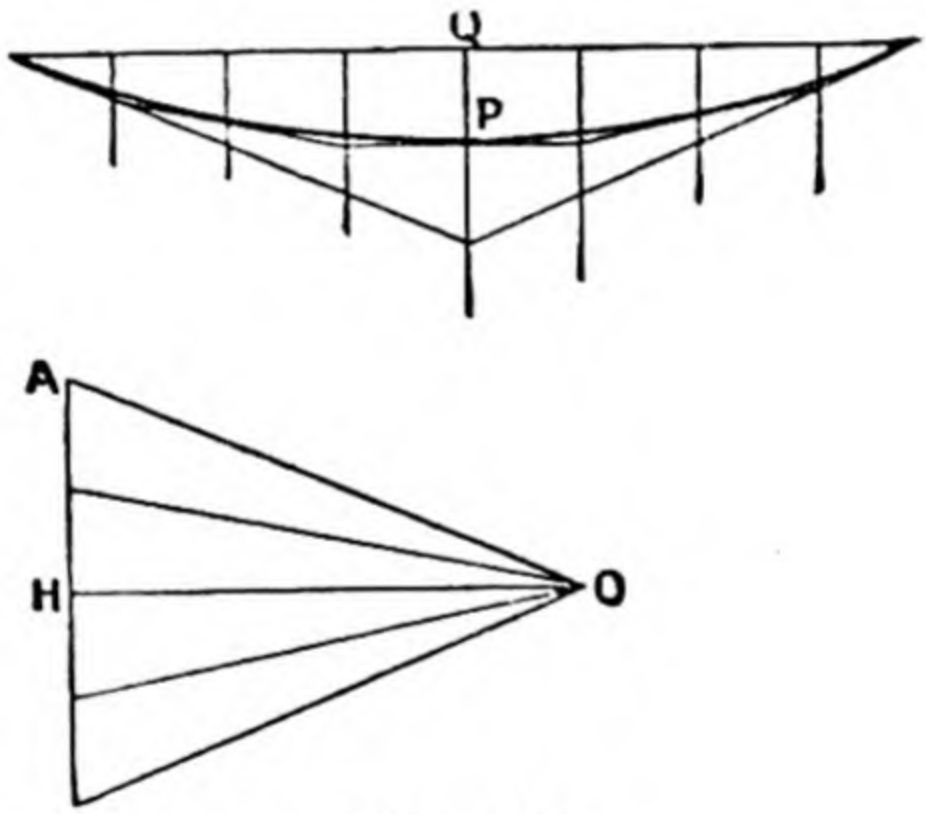


FIG. 140.

as four equal parts, and the approximate true curve for a uniform string stretched with a moderate tension, are shown in Fig. 140.

Note that the dip QP would be less if the tensions OH, OA, etc., were increased.

The relations between the dip, weight, and tension of a stretched string or chain, assuming perfect flexibility, can more conveniently be found by ordinary calculation than by graphical methods.

Assuming that the dip is small and the load per horizontal foot of span is uniform throughout, the equilibrium of a

portion AP (Fig. 141) of horizontal length x, measured from the lowest point A, may be considered.

Let w = weight per unit horizontal length of cord or chain;

y = vertical height of P above A, viz. PQ (Fig. 141);

T = the tension (which is horizontal) at A;

T′ = the tension at P acting in a line tangential to the curve at P.

The weight of portion AP is then wx, and the line of

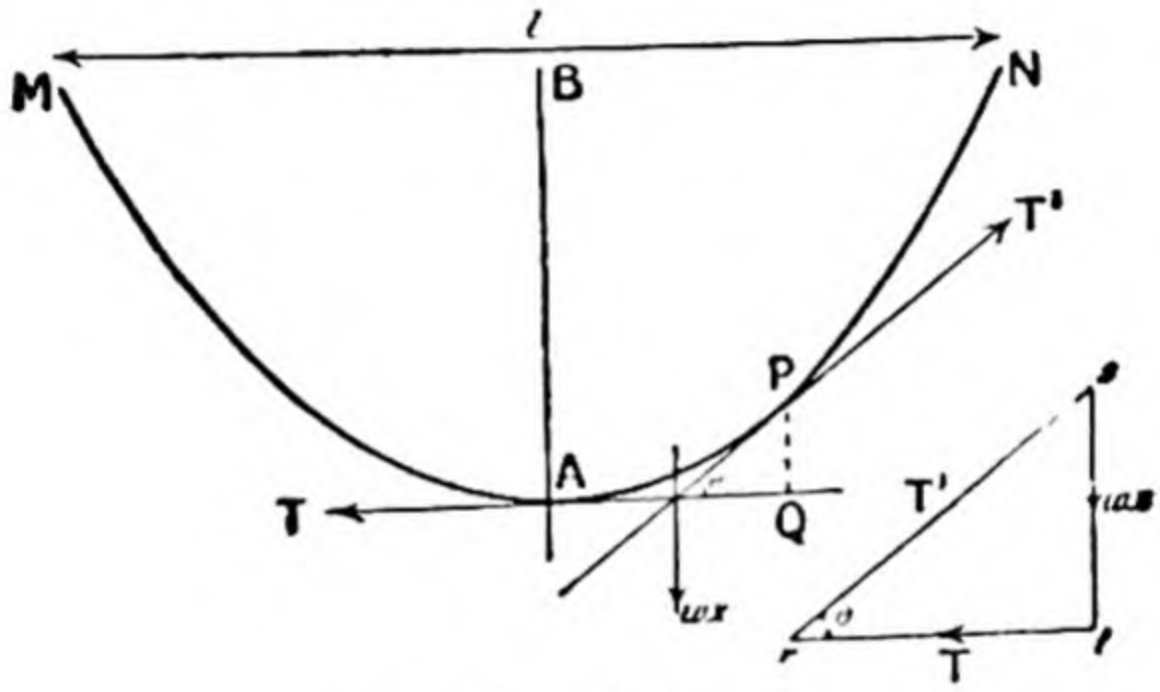


FIG. 141.

action of the resultant weight is midway between AB and PQ, *i.e.* at a distance $x/2$ from either.

Taking moments about the point P

$$T \times PQ = wx \times x/2 \quad \text{or} \quad T \times y = wx^2/2$$

$$y = wx^2/2T$$

This relation shows that the curve of the string is a parabola.

If d = the total dip AB, and l = the span of the string or chain, taking moments about N of the forces on the portion AN and transposing,

$$d = \frac{w(l/2)^2}{2T} = \frac{wl^2}{8T} \quad \text{or} \quad T = \frac{wl^2}{8d}$$

which gives the relation between the dip, the span, and the horizontal tension.

Returning to the portion AP, if the vector triangle *rst* be drawn for the forces acting upon it, the angle θ which the tangent to the curve at P makes with the horizontal, is given by the relation

$$\frac{xw}{T}=\frac{st}{tr}=\tan\theta$$

Also the tension T′ at P is T sec θ, or

$$T'=T\sqrt{1+\tan^2\theta}=T\sqrt{1+(wx/T)^2}=\sqrt{T^2+w^2x^2}$$

and at the ends, where $x=l/2$,

$$T'=\sqrt{T^2+w^2l^2/4}$$

And since $T=wl^2/8d$, the tension at N or M is

$$\frac{wl}{2}\sqrt{\frac{l^2}{16d^2}+1},\quad \text{or}\quad \frac{wl^2}{8d}\sqrt{1+\frac{16d^2}{l^2}}$$

which does not greatly exceed $wl^2/8d$ (or T), if d/l is small.

EXAMPLE. A copper trolley-wire weighs ½ lb. per foot length; it is stretched between two poles 50 feet apart, and has a horizontal tension of 2000 lb. Find the dip in the middle of the span.

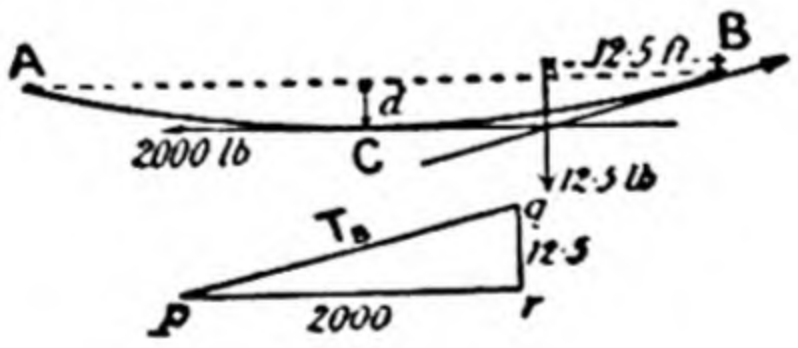


FIG. 142.

Let d=the dip in feet.

The weight of the wire in the half-span BC (Fig. 142) is $25\times\frac{1}{2}=12{\cdot}5$ lb.

The distance of the c.g. of the wire BC from B is practically 12·5 feet horizontally.

Taking moments about B of the forces on the portion BC,

$$2000\text{ lb.}\times d=12{\cdot}5\text{ lb.}\times 12{\cdot}5\text{ ft.}$$

$$d=0{\cdot}07812\text{ foot}=0{\cdot}937\text{ inch.}$$

EXAMPLES XI

1. A roof principal, shown in Fig. 143, carries loads of 4, 7, and 5 tons in the positions shown. It is simply supported at the extremities of a span of 40 feet. The total rise of the roof is 14 feet, and the distances PQ and RS are each 5·4 feet. Draw the stress diagram and find the stress in each member of the frame.

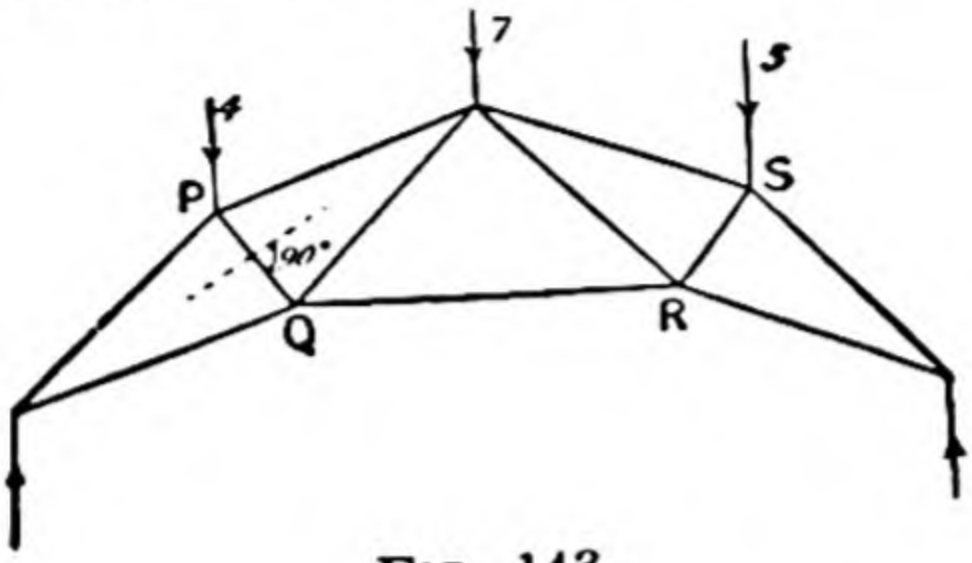


FIG. 143.

2. A Warren girder (Fig. 144), made up of bars of equal lengths, carries a single load of 5 tons as shown. Draw the stress diagram and scale off the forces in each member; check the results by the method of sections (Art. 28).

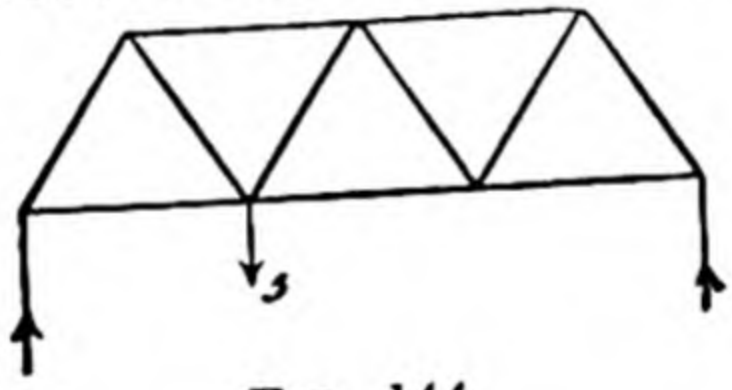


FIG. 144.

3. Draw the stress diagram for the roof-frame in Fig. 145

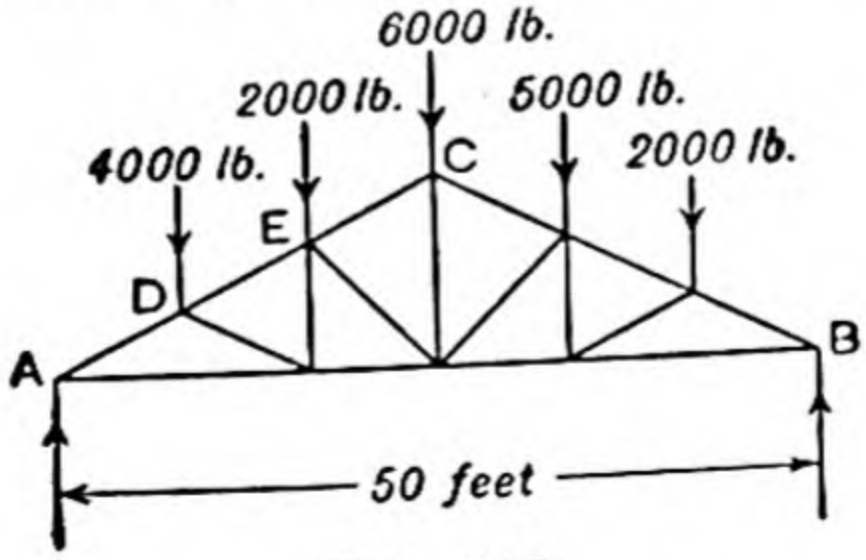


FIG. 145.

under the given loads. The main rafters are inclined at 30° to the horizontal, and are each divided by the joints into three equal lengths.

4. In place of the loads shown in Fig. 145, the frame has at D, E, and C, loads of 2000 lb., 2000 lb., and 1000 lb. respectively, all perpendicular to AC. Find the supporting forces at A and B if A is hinged and B is supported on rollers. Draw the stress diagram.

5. A chain connects two points on the same level and 10 feet apart; it has suspended from it four loads, each of 50 lb., at equal horizontal intervals along the span. If the tension in the middle section is 90 lb., draw the shape of the chain, measure the inclination to the horizontal, and the tension of the end section.

6. Find the shape of a string connecting two points 8 feet horizontally apart, one being 1 foot above the other, when it has suspended from it weights of 5, 7, and 4 lb. at horizontal distances of 2, 5, and 6 feet respectively from the higher end, the horizontal tension of the string being 6 lb.

7. A light chain connects two points, X and Y, 12 feet horizontally apart, X being 2 feet above Y. Loads of 15, 20, and 25 lb. are suspended from the chain at horizontal distances of 3, 5, and 8 feet respectively from X. The chain passes through a point 7 feet horizontally from X and 4 feet below it. Draw the shape of the chain. How far is the point of suspension of the 15-lb. load from X?

8. A wire is stretched horizontally, with a tension of 50 lb., between two posts 60 feet apart. If the wire weighs 0·03 lb. per foot, find the sag of the wire in inches.

9. A wire weighing 0·01 lb. per foot is stretched between posts 40 feet apart. What must be the tension in the wire in order to reduce the sag to 2 inches?

10. A wire which must not be stretched with a tension exceeding 70 lb. is to be carried on supporting poles, and the sag between two poles is not to exceed 1·5 inches. If the weight of the wire is 0·025 lb. per foot, find the greatest distance the poles may be placed apart.

CHAPTER XII

BENDING AND TWISTING

109. Beams and Bending. In the preceding chapter we considered the stresses in structural members which were subjected only to loads directed along the line of their longitudinal axes. But frequently parts of a structure or of a machine sustain loads perpendicular to their axes. Such a part is called a beam and its strain or deformation takes the form of bending or flexure, although the bending may be too small to be visible to the eye unassisted by an instrument. We now proceed to consider the stress produced in a beam by the straining actions of external forces.

Beams are frequently horizontal and the external forces exerted upon them vertical, and it will be convenient to speak of beams as horizontal and external forces as vertical although in fact they need not be so.

110. Straining Actions on Beams. Shearing Force and Bending Moment. Before investigating the stresses and strains set up in bending, the straining actions resulting from various systems of loading and supporting beams will be considered.

If we consider a beam carrying a number of transverse

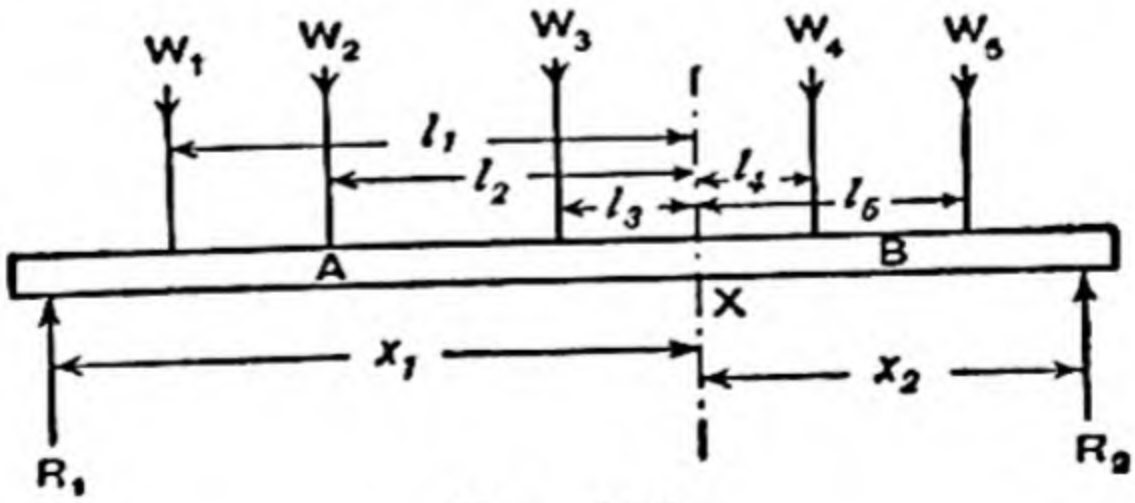


FIG. 146.

loads, as in Fig. 146. the whole beam is in equilibrium under the action of the loads W_1, W_2, W_3, etc., and the

supporting forces or reactions R_1 and R_2; further, if we divide the beam into two parts, A and B, by an ideal plane of section X, each part is in equilibrium under the action of forces, and the conditions of equilibrium (Art. 26) apply to the forces on either part, provided we include in them the forces which the other part exerts on it across the section X. Thus the system of forces which keeps the part A in equilibrium consists of W_1, W_2, W_3, and R_1, together with the forces exerted on A by B across the section X in virtue of the state of stress in the beam. We may conveniently consider these latter forces by estimating their total horizontal and vertical components and their moments. We apply the three ordinary conditions of equilibrium, Art. 26, and draw three conclusions.

(1) Since the algebraic sum of the vertical forces on A is zero, the total upward force exerted by B on A must be numerically equal to the downward external force on A, viz. $W_1+W_2+W_3-R_1$.

This transverse force exerted upward by B on A (and downward by A on B) is called the *shearing force* at the section X and may be denoted by the symbol F; and at X,

$$F=W_1+W_2+W_3-R_1 \text{ or } R_2-(W_4+W_5) . \quad (1)$$

since $R_1=W_1+W_2+W_3+W_4+W_5-R_2$. Thus the shearing force at X is equal to the algebraic sum of the vertical forces on either side of X.

(2) Since there are no horizontal forces acting on the piece A except those across the section X, the algebraic total horizontal component of those forces is zero, or the total pull is numerically equal to the total thrust.

(3) The moment, about X say, of the vertical forces exerted on A must be balanced by the moment of the forces which B exerts on A. The moment of the external vertical forces is called the *bending moment*, and the equal and opposite moment of the (internal) forces which B exerts on A is called the moment of resistance at the section X which separates A and B. Both these equal and opposite moments are denoted by the symbol M. Thus, using the notation of Fig. 146, clockwise,

$$M=R_1x_1-W_1l_1-W_2l_2-W_3l_3 . . . \quad (2)$$

and since A and B exert equal and opposite moments on one another, from the equilibrium of piece B, similarly,

$$M = W_4 l_4 + W_5 l_5 - R_2 x_2$$

Thus the bending moment M is the algebraic sum of the moments of all the external forces on either side of the section considered. The conditions of equilibrium show that the moment of resistance and the bending moment are numerically equal and of opposite kind or sign.

Signs. The signs of F and M are conventional and any convention may be adopted. At this stage and for simple loading and supports, the signs are not very important. We have here in (1) taken F as positive at any section when the algebraic sum of the downward external forces to the left of that section is positive. Or when the algebraic sum of the upward external forces to the right of the section is positive.

The sign of M has been chosen so that it is positive at any section when the moment of the external forces on the portion to the left of that section is clockwise. (The moment of resistance or moment of internal forces exerted by B on A at X is then opposite, *i.e.* counter-clockwise.) Here the convention of signs is unimportant, but notice must be taken of changes of sign if they occur, at different distances along a beam.

111. Diagrams of Shearing Force and Bending Moment. In order to be able to calculate the intensity of stress at any part of the material of a beam it is necessary to know the straining actions at every section, *i.e.* the shearing force and bending moment. Both will generally vary in magnitude from point to point along the length of a loaded beam; their values at any given cross-section can often be calculated arithmetically, or general algebraic expressions may give the bending moment and shearing force for any section along the beam. The variation may also be shown graphically by plotting graphs the bases of which represent to scale the length of the beam, and the vertical ordinates the bending moments or shearing forces, as the case may be. Some simple typical examples of bending moment and shearing forces curves follow in Figs. 147 to 152, inclusive. In each case, M represents bending moment, F shearing

force, and R reaction or supporting force, with appropriate suffixes to denote the positions to which the letters refer.

Fig. 147 refers to the very simple case of a cantilever (*i.e.* of a beam firmly fixed at one end and free from any other support) carrying a single concentrated load W at its free end. At any point X distant x from the free end, the shearing force (F) is equal to the load W or —W according to which sign we adopt. The shearing force is constant throughout the length (l) of the cantilever, and the shearing force diagram is a line parallel to the axis of the beam and distant W from the base line of the

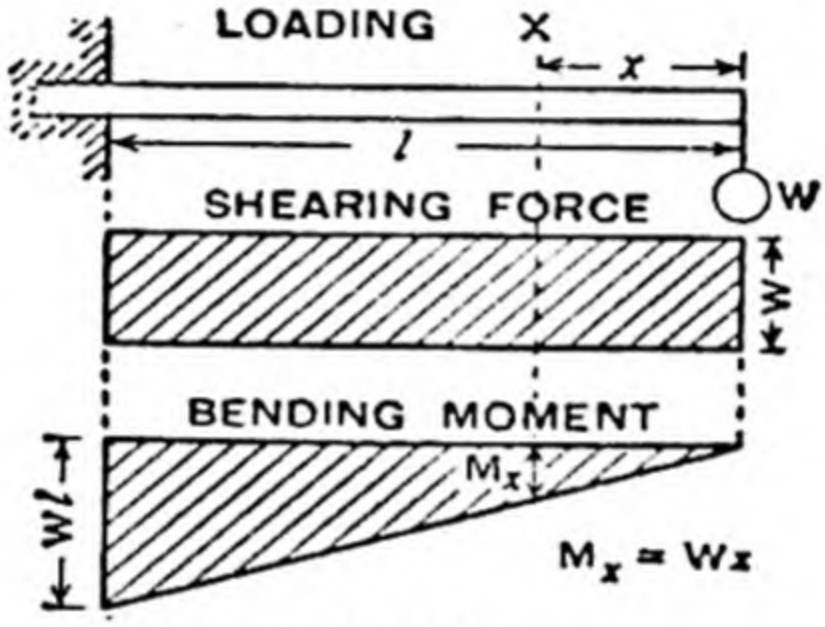


FIG. 147.

diagram. The bending moment at X is Wx, which is the clockwise moment of the external force (W) to the right of X, and the bending moment diagram is a line sloping from zero value at the free end, where $x=0$, to a maximum Wl at the fixed (or wall) end where $x=l$.

Fig. 148 shows the diagrams for a cantilever with three concentrated loads W_1, W_2, and W_3, distant l_1, l_2, and l_3 respectively from the fixed end. Here the shearing force increases abruptly or discontinuously at each point where a concentrated load is exerted. And at these same points the slope or rate of change of the bending moment also changes abruptly. Between the loads W_1 and W_2, the B.M. increases at a rate W_1 per unit length (or W_1x in x units of length). Between the loads W_2 and W_3, the B.M. increases at the rate $(W_1+W_2)x$ for x units of length. Notice that the magnitude (apart from sign) of the rate of increase of bending moment at any section is equal to the

shearing force at that section. Also that the B.M. is equivalent to three B.M. diagrams (like that of Fig. 147) superposed so that their ordinates are added. These

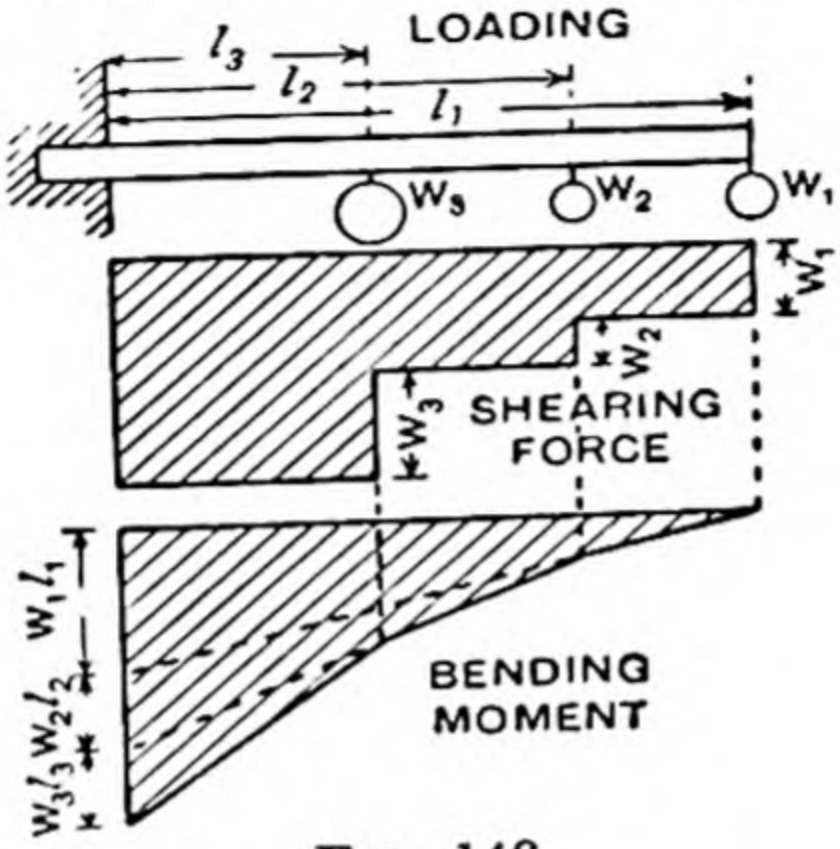


FIG. 148.

three separate diagrams would rise from zero at the loading points to $W_1 l_1$, $W_2 l_2$, and $W_3 l_3$ at the fixed end of the cantilever. This principle of adding the ordinates of

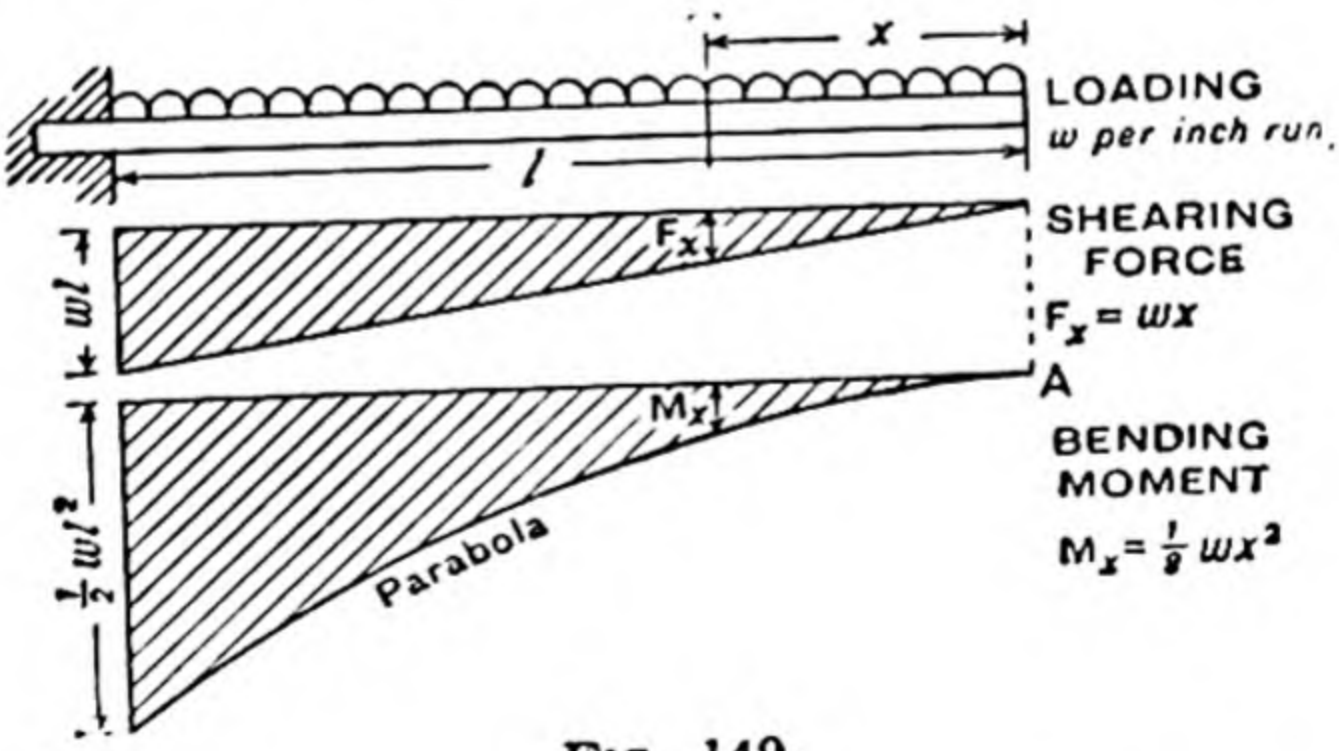


FIG. 149.

B.M. diagrams of separate loads is often useful when there are several loads on beams of all kinds and not only for cantilevers.

Fig. 149 shows the diagrams for a cantilever with a

uniformly distributed load of w per unit length, say, per inch if the lengths are in inches and B.M. in pound-inches. At a section X, distant x from the free end, the total load to the right of the section is wx, and this is the shearing force. The S.F. diagram is a straight line and the maximum shearing force is wl at the fixed end, this being the total load on the beam. The B.M. at the section X denoted by M_X is that of a force wx with its c.g. $x/2$ from the section, so that $M_X = wx \times x/2 = \frac{1}{2}wx^2$, reaching a maximum of $\frac{1}{2}wl^2$ when $x = l$ at the fixed end.

Fig. 150 shows the case of the beam simply supported at its ends and carrying a concentrated load W midway between the ends. The supporting force at each end is

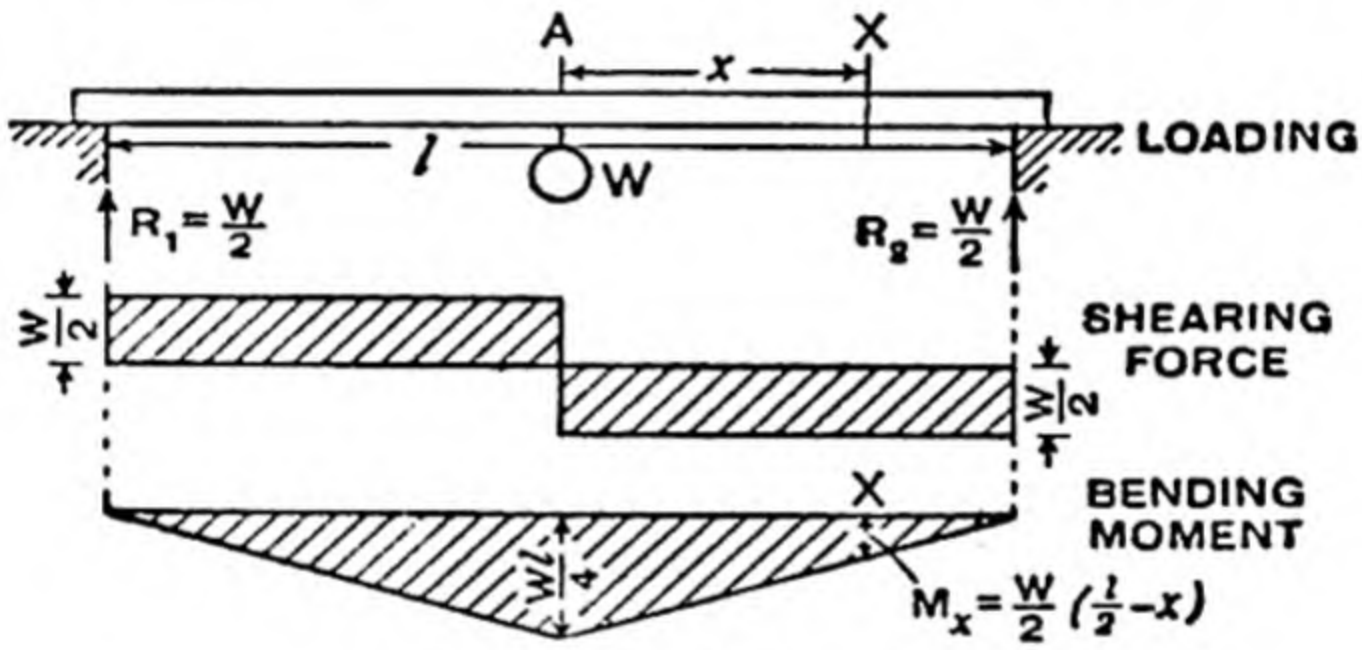


FIG. 150.

$\frac{1}{2}$W, and this is the magnitude of the S.F. from either support to the centre of the span, where it changes by the force W of opposite sign to $\frac{1}{2}$W of opposite sign. The B.M. at a distance x from the centre (*i.e.* $\frac{1}{2}l - x$ from the right-hand support) is $M = R_2(\frac{1}{2}l - x) = \frac{1}{2}W(\frac{1}{2}l - x)$, provided x is not greater than $\frac{1}{2}l$, *i.e.* in the right-hand half of the span. It reaches a maximum $\frac{1}{4}Wl$ at the centre A. In the other half of the span, the B.M. varies in the same way proportionally to the distance from the left-hand support, from zero to $\frac{1}{4}Wl$.

Fig. 151 shows the more general case, where the load is not at midspan but distant a from the left support and b from the right support, so that $a + b = l$. Here $R_1 = W \times b/l$ and $R_2 = W \times a/l$. The B.M. rises from zero at either support to Wab/l at the load, following straight lines

for intermediate points; while the S.F. changes from R_1, or Wb/l, at the left-hand support, by an amount of W of

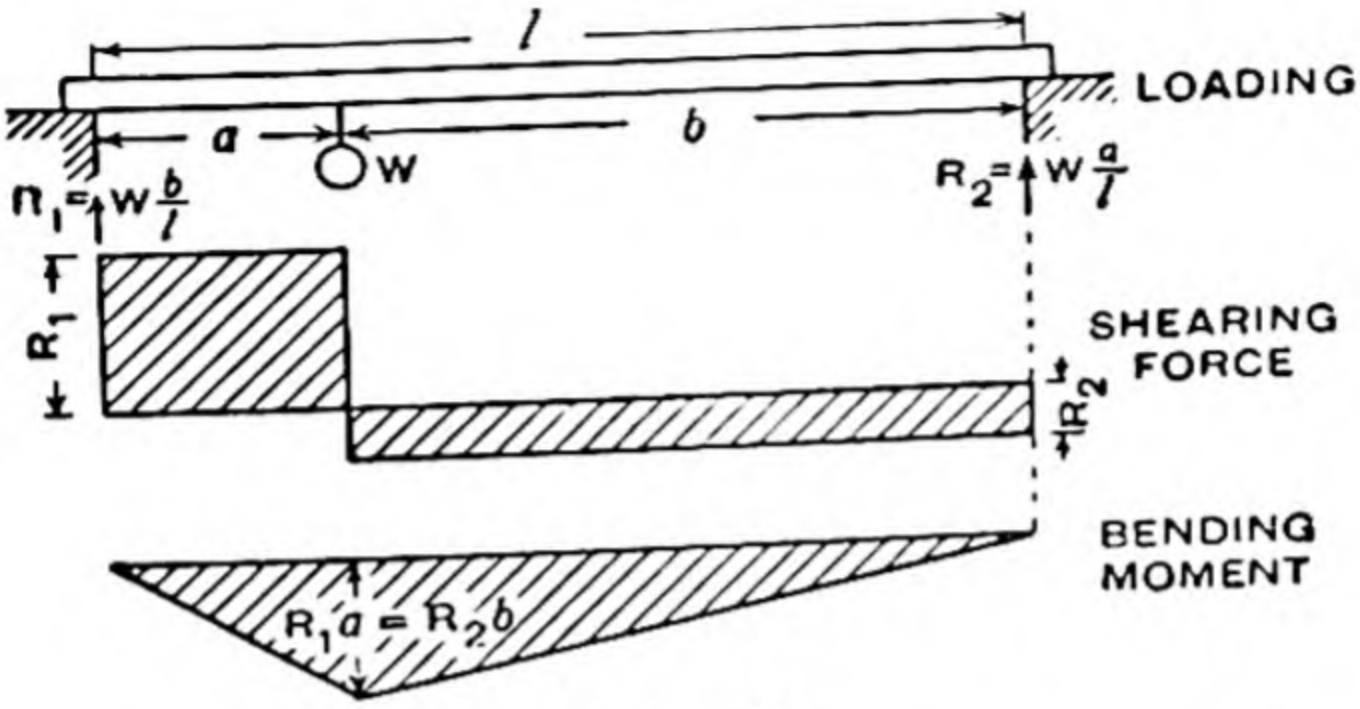


FIG. 151.

opposite sign at the load to Wa/l of opposite sign for the remainder of the span.

Fig. 152 shows the diagrams for a beam of span l simply supported at the ends of the span and carrying a uniformly

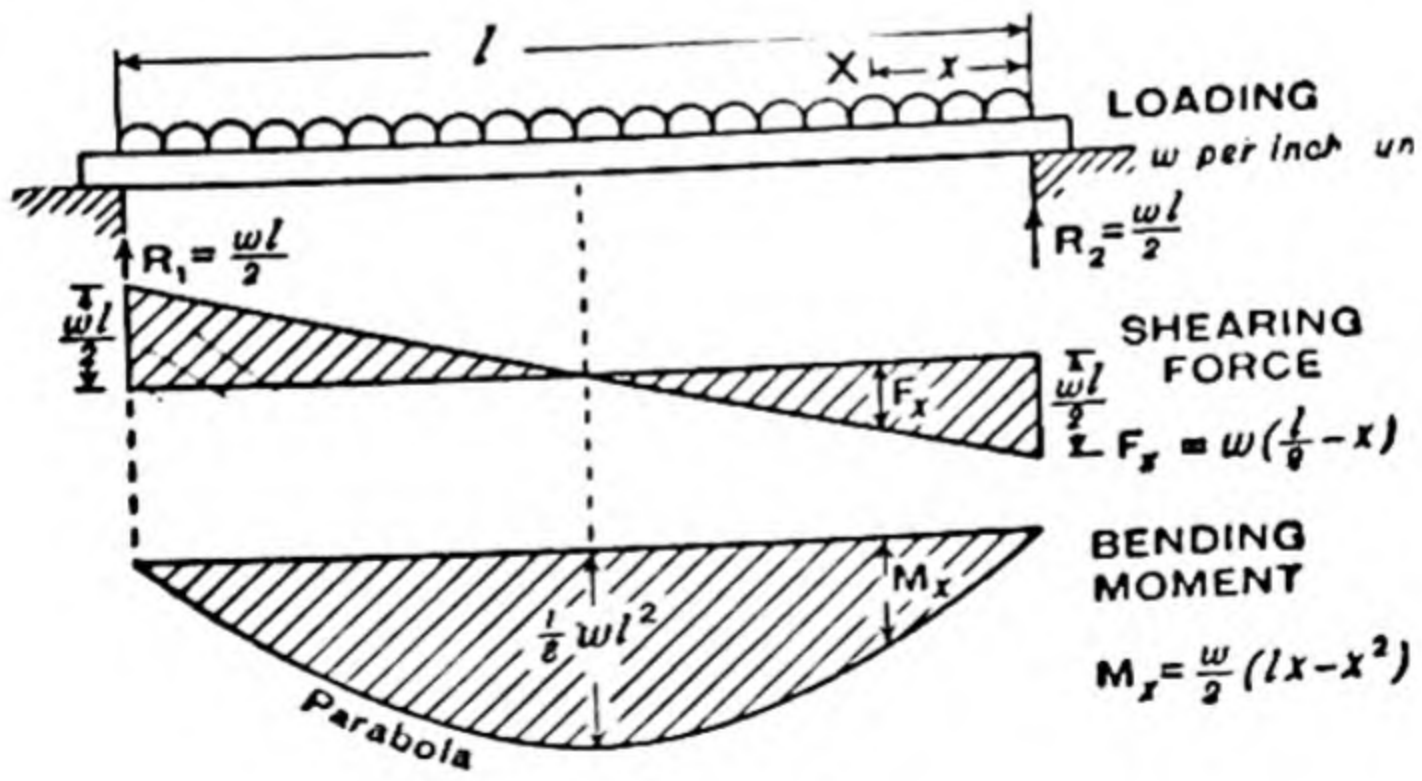


FIG. 152.

spread load w per unit length. The end supporting forces are each $\frac{1}{2}wl$. At a distance x from the right end, the upward force is $\frac{1}{2}wl$ and the downward force wx, giving a

net upward force to the right of X of $\frac{1}{2}wl - wx$ or $F_X = w(\frac{1}{2}l - x)$. Here with no abrupt or discontinuous load the expression holds good throughout the span, but of course its numerical value changes sign at $x = \frac{1}{2}l$, *i.e.* at the middle of the span where $F = 0$. The B.M. at X, from moments to the right of X, is

$$M_X = R_2 x - wx \times \tfrac{1}{2}x = \tfrac{1}{2}w(lx - x^2) \quad \text{or} \quad \tfrac{1}{2}wx(l - x)$$

which is a parabola reaching a maximum value $wl^2/8$ when $x = \frac{1}{2}l$, *i.e.* midway between the supports.

In each of the foregoing cases it is to be noticed that the rate of change of bending moment is equal to the S.F. Where the S.F. is constant, the B.M. diagram or graph follows a straight line of constant slope. Where the S.F. $= 0$, the slope of the B.M. diagram is zero or passes discontinuously through zero with change of sign. Where, as in Figs. 149 and 152, the S.F. varies continuously the slope of the B.M. diagram is not constant but varies continuously.

The foregoing diagrams by algebraic expressions illustrate important relationships and also provide formulæ for common or standard cases of loading. But in other, or in any, cases of simple support, the B.M. and S.F. can generally be calculated by arithmetic for given numerical data, which may be illustrated by examples.

EXAMPLE 1. A beam freely supported at its ends 20 feet apart, carries loads of 4, 3, and 5 tons at distances of 4, 9, and 15 feet respectively from the left-hand end. Find the supporting forces, or reactions, at its ends and draw the shearing force and bending moment diagrams. Where must an additional load of 2 tons be placed if the two supporting forces are to be equal ?

The loading is shown at the top of Fig. 153. To find R_B, the reaction at B, equate the opposite moments about A.

$$R_B \times 20 \text{ ft.} = 4 \text{ tons} \times 4 \text{ ft.} + 3 \text{ tons} \times 9 \text{ ft.} + 5 \text{ tons} \times 15 \text{ ft.}$$

$$= (16 + 27 + 75) \text{ ton-ft.} = 118 \text{ ton-ft.}$$

$$R_B = 118 \text{ ft.-tons}/20 \text{ ft.} = 5{\cdot}9 \text{ tons}$$

$$R_A = (4 + 3 + 5 - 5{\cdot}9) \text{ tons} = 6{\cdot}1 \text{ tons.}$$

Shearing Forces.

A to C, $F = R_A = 6{\cdot}1$ tons

C to D, $F = 6{\cdot}1 - 4 = 2{\cdot}1$ tons

D to E, $F = 6{\cdot}1 - 4 - 3 = -0{\cdot}9$ ton

E to B, $F = 6{\cdot}1 - 4 - 3 - 5 = -5{\cdot}9$ tons (or R_B).

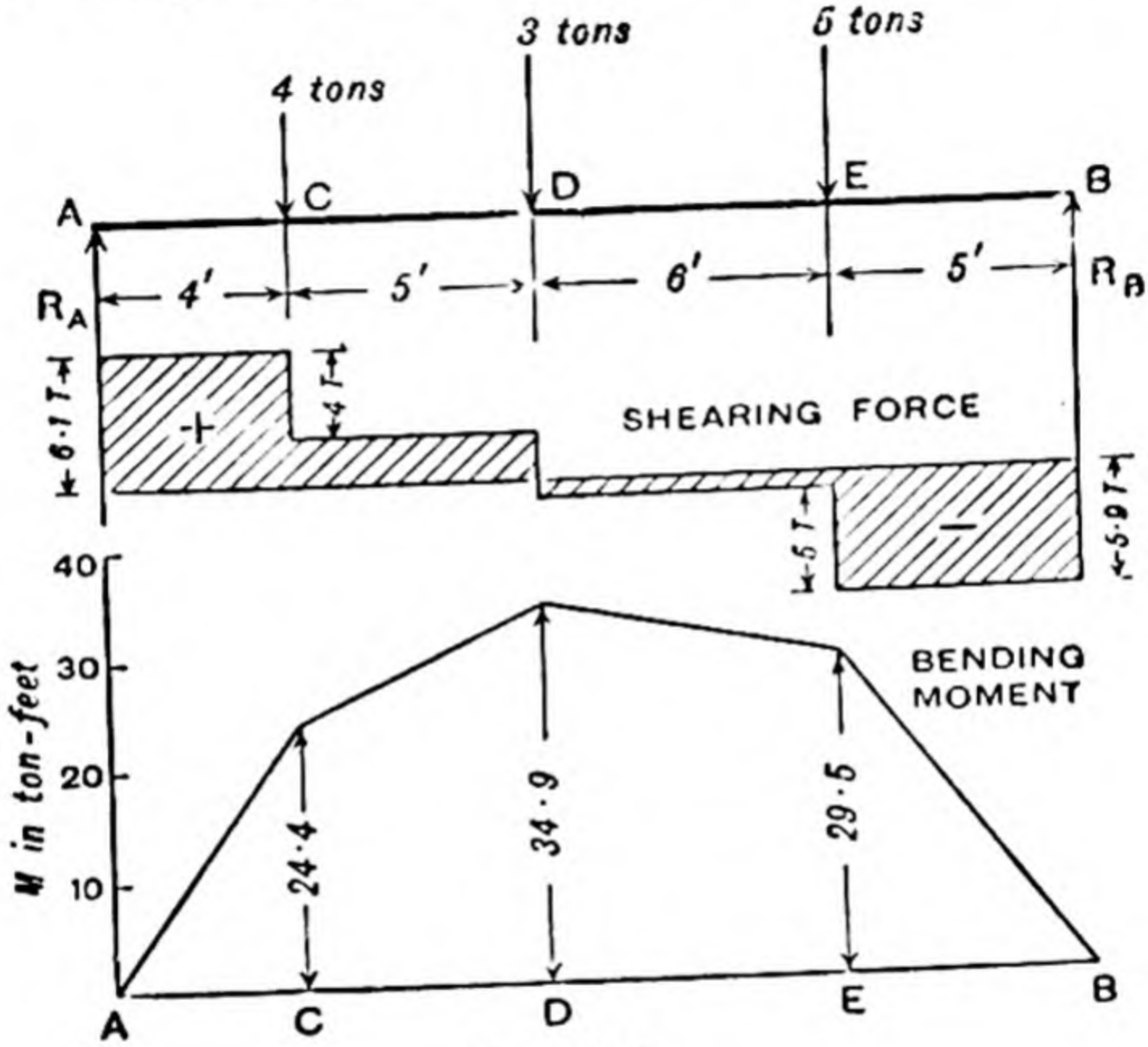


FIG. 153.

The S.F. diagram is shown in Fig. 153. We have here made S.F. positive at any section when the upward forces to the left of the section are positive.

Bending moments.

$M_C = R_A \times 4 = 6{\cdot}1$ tons $\times 4$ ft. $= 24{\cdot}4$ ton-ft.

$M_D = 6{\cdot}1 \times 9 - 4 \times 5 = 54.9 - 20 = 34{\cdot}9$ ton-ft.

$M_E = 6{\cdot}1 \times 15 - 4 \times 11 - 3 \times 6 = 29{\cdot}5$ ton-ft.

or to the right of E

$M_E = 5{\cdot}9$ tons $\times 5$ ft. $= 29{\cdot}5$ ton-ft.

The B.M. diagram is shown in Fig. 153 and the maximum B.M. is at D, viz. 34·9 ton-ft.

For equal supporting forces with an additional load of 2 tons, let x be the distance of this load from A. Then the total load is 14 tons and each supporting force is 7 tons. Equating opposite moments about A,

$$7 \text{ tons} \times 20 \text{ ft.} = 4 \text{ tons} \times 4 \text{ ft.} + 3 \text{ tons} \times 9 \text{ ft.} + 5 \text{ tons} \times 15 \text{ ft.} + 2 \text{ tons} \times x$$

$$2x \text{ tons} = 140 - 118 \text{ ton-ft.} = 22 \text{ ton-ft.}$$

$$x = 22 \text{ ton-ft.}/2 \text{ tons} = 11 \text{ ft.}$$

The maximum B.M. will now be at D or under the 2 ton load (these being most central).

$$M_D = 7 \text{ tons} \times 9 \text{ ft.} - 4 \text{ tons} \times 5 \text{ ft.} = 43 \text{ ton-ft.}$$

At 11 ft. from A,

$$M = 7 \text{ tons} \times 11 \text{ ft.} - 4 \text{ tons} \times 7 \text{ ft.} - 3 \text{ tons} \times 2 \text{ ft.}$$
$$= 77 - 34 = 43 \text{ ton-ft.}$$

The B.M. is constant between these two sections, the B.M. diagram being a horizontal line and the S.F. being zero. ($F = 7 - 4 - 3 = 0$ or $F = 7 - 2 - 5 = 0$.)

EXAMPLE 2. A beam 20 feet long is freely supported 3 feet from the left end and 5 feet from the right end. It carries loads of 4, 7, and 5 tons at distances of 0, 6, and 10 feet respectively from the left end. Find what load must be carried at the right end in order that the two supporting forces shall be equal and then draw the B.M. and S.F. diagrams.

The beam and loads are shown at the top of Fig. 154, W being the unknown load at B.

$$\text{Total load} = 4 + 7 + 5 + W = 16 + W$$

$$R_C = R_D = \tfrac{1}{2}(16 + W) = \tfrac{1}{2}W + 8 \text{ tons}$$

Equating opposite moments about C, since the beam is in equilibrium,

$$7 \text{ tons} \times 3 \text{ ft.} + 5 \text{ tons} \times 7 \text{ ft.} + W \times 17 \text{ ft.}$$
$$= (\tfrac{1}{2}W + 8)12 + 4 \times 3$$

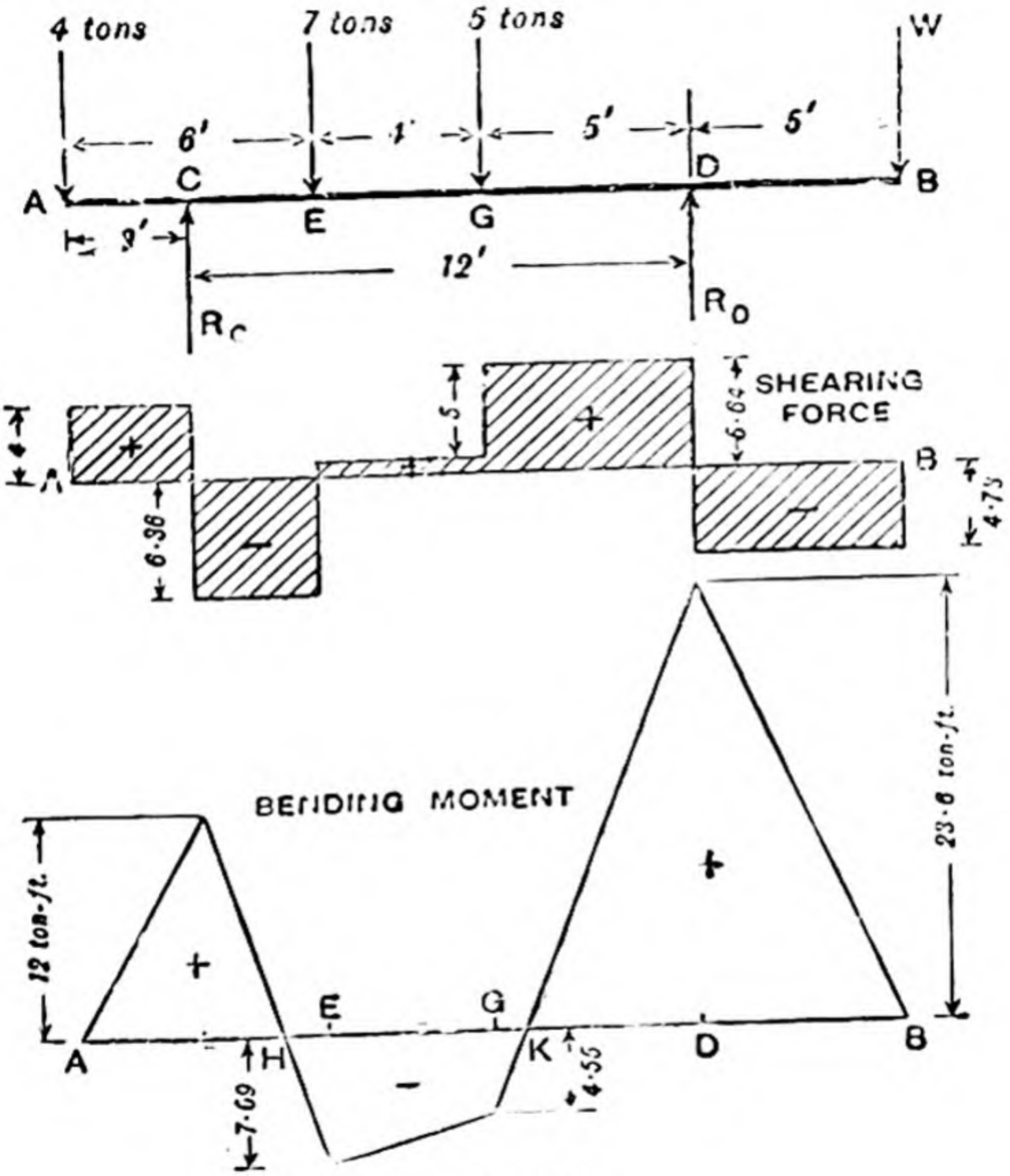


FIG. 154.

$$56 + 17W = 6W + 96 + 12$$
$$11W = 52$$
$$W = 4{\cdot}7272 \text{ tons}$$
$$R_C = R_D = 10{\cdot}364 \text{ tons}$$

Shearing Forces.

A to C	F = 4 tons
C to E	F = 4 − 10·364 = −6·36 tons
E to G	F = 11 − 10·36 = 0·64 ton
G to D	F = 16 − 10·36 = 5·64 tons
D to B	F = 5·64 − 10·36 = −4·72 (= −W)

The S.F. diagram is shown in Fig. 154. We have in this case made S.F. positive when the *downward* forces to the left of any section are positive.

Bending moments.

$M_C = 4$ tons $\times 3$ ft. $= 12$ ton-ft.

$M_E = 4 \times 6 - 10{\cdot}364 \times 3 = -7{\cdot}09$ ton-ft.

$M_G = 4 \times 10 + 7 \times 4 - 10{\cdot}364 \times 7 = -4{\cdot}55$ ton-ft

$M_D = 4 \times 15 + 7 \times 9 + 5 \times 5 - 10{\cdot}36 \times 12 = 23{\cdot}64$ ton-ft.

check,

$M_D = 5 \times 4{\cdot}7272 = 23{\cdot}64$ ton-ft.

$M_B = 0$.

The B.M. has here been taken as positive at any section when the moment of external forces to the left of that section have a counter-clockwise moment about it. In Example 1 we adopted the opposite convention of signs. We have a free choice and physically there is nothing essentially negative about any moment. But in this example the B.M. changes sign at the sections corresponding to the points H and K. Between these points the beam is bent concave upwards, and outside of them convex upwards.

112. Bending Moments and Shearing Forces from Funicular and Vector Polygons. The vertical breadths of a funicular or link polygon for a system of vertical forces on a horizontal beam represent to scale the bending moments at the corresponding sections. This has already been proved in Art. 35, for it was shown that in Fig. 22 the moment of R_1, W_1, and W_2 at section X is represented by the intercept xl

on the funicular polygon. This polygon *pqlmwz* on the sloping base *zp* is in fact the B.M. diagram for the beam with the given loads. This is further illustrated in Fig. 155, where, after the reactions R_1 and R_2 have been determined by a trial funicular polygon with any pole or by arithmetical calculation, the funicular polygon shown has been drawn on a horizontal base by making the vector *fo* in the

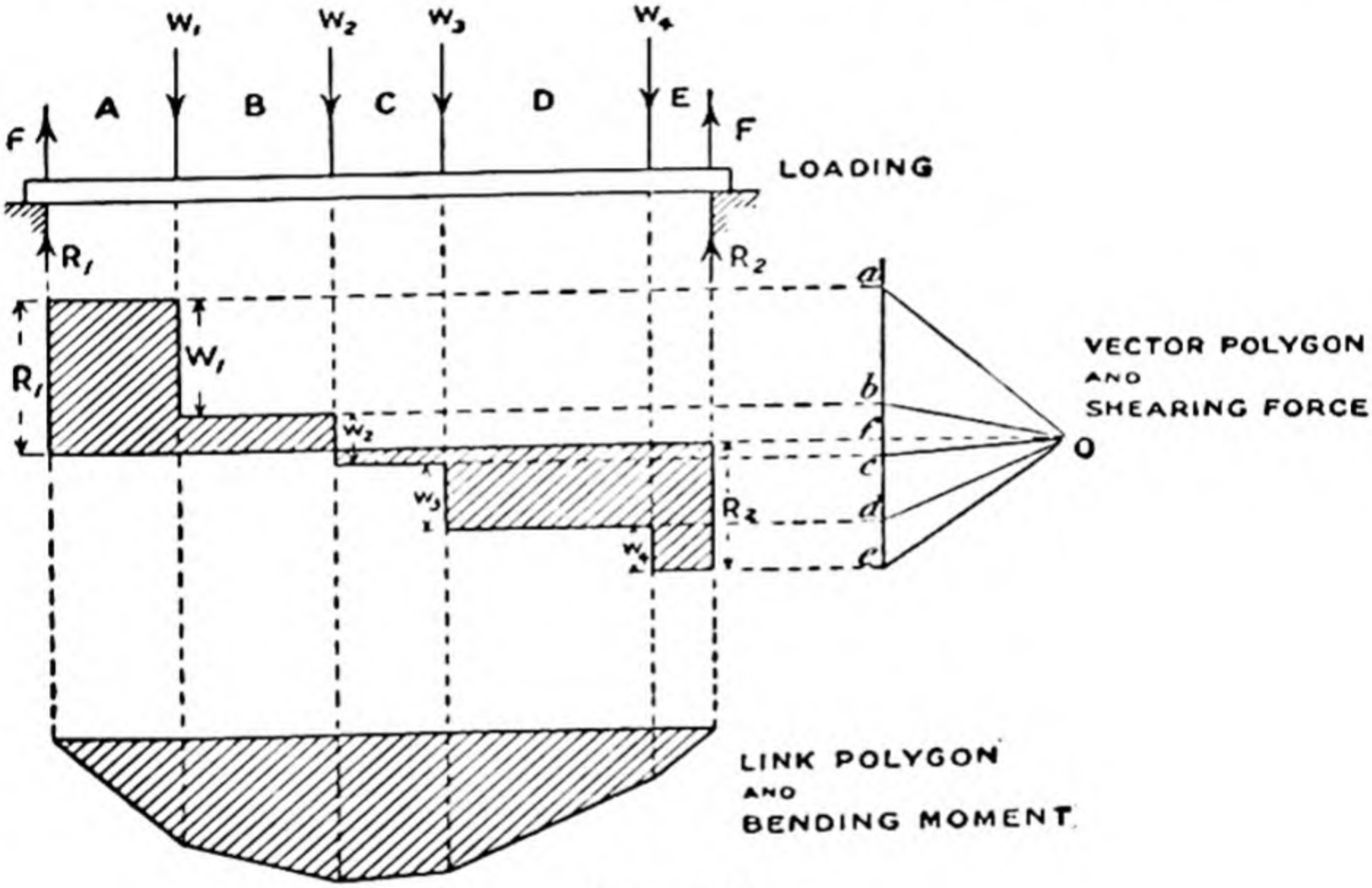


FIG. 155.

vector polygon horizontal, *i.e.* by choosing a pole *o* in the same horizontal line as the point *f*, which divides the load-line *abcde* in the ratio of the supporting forces. The position of *f* can be calculated or found by means of a trial link polygon with any pole. The scale of bending moment, as explained in Art. 34, is $p \cdot q \cdot h$ lb.-inches to 1 inch where the scale of force is p lb. to 1 inch, of distance q inches to 1 inch, and the pole distance *fo* measures h inches. It is not necessary to draw the diagram on a horizontal base, but the distance h must be estimated horizontally, and the ordinates of bending moment must be measured vertically.

The shearing-force diagram is shown projected from the vertical load-line of the vector polygon. Thus the S.F. across the length corresponding to the space A is R_1 represented by fa; for space B the S.F. is $R_1 - W_1$ represented by $fa - ab = fb$ and so on.

The same method of drawing the bending-moment diagram to as close approximation as is desired is applicable to loads distributed either uniformly or otherwise by

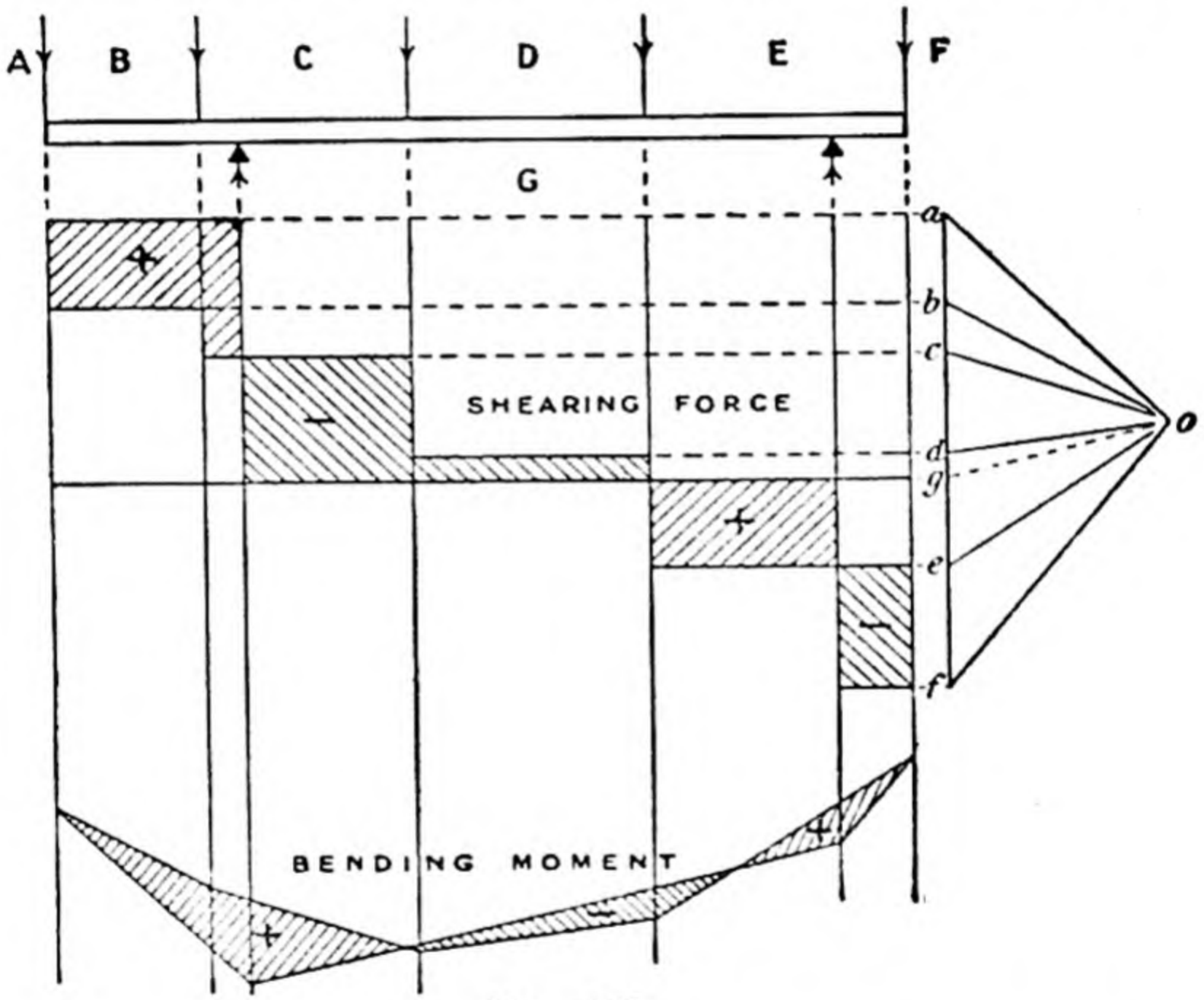


FIG. 156.

dividing the load into a number of sections along the length of the beam, and treating each part as a load concentrated at its centre of gravity. The resulting funicuiar polygon will be a figure with straight sides, and the curve of bending moments is the *inscribed* (not *circumscribed*) curve touching the sides of the polygon, for the polygon evidently gives excessive ordinates at the points of concentration and correct ones at the junctions of the parts into which the

loaded lengths are divided. Consideration with a sketch of an extreme case, say a uniform load throughout the span and only two equal divisions, will make this clear.

Fig. 156 shows the bending moment diagram for a beam with overhanging ends. The reactions are found as in Fig. 21, the most convenient order of lettering and setting off the forces on the vector polygon being consecutively round the beam. Care is then required in projecting the shearing-force diagram, as the forces do not follow in consecutive order across the paper. It will be found instructive to redraw the vector polygon in such consecutive order and project a shearing-force diagram from it. The choice of signs in Fig. 156 is, of course, arbitrary. Those shown are in accordance with the convention used in Example 2, Art. 111, and Fig. 154.

113. Theory of Elastic Bending. The relations existing between the straining force, the dimensions, the stresses, strains, elasticity, and curvature of a beam are under certain simple assumptions very easily established for the case of *simple bending*, *i.e.* flexure by pure couples applied to a beam without shearing force.

Most of the same simple relations may generally be used as close approximations in cases of flexure which are not "simple," but which are of far more common occurrence, the strains involved from the shearing force being negligible.

A straight bar of homogeneous material, subjected only to equal and opposite couples at its ends, has a uniform bending moment throughout its length, and if there is no shearing force, is said to suffer simple bending. The beam will be supposed to be of the same cross-section throughout its length, and symmetrical about a central longitudinal plane, in and parallel to which bending takes place. In Fig. 157, central longitudinal sections before and after bending and a transverse section are shown, the cross-section being symmetrical about an axis YY.

It will be assumed that transverse plane sections of the beam remain plane and normal to longitudinal fibres after bending, which is reasonable since the straining action is the same on every section.

Consider any two transverse sections AB and CD very close together. After bending, as shown at A′B′ and C′D′

they will not be parallel, the layer of material at AC being extended to A′C′, and that at BD being pressed to B′D′. The line EF represents the layer of material which is neither stretched nor shortened during bending. This surface EF suffers no longitudinal strain, and is called the *neutral*

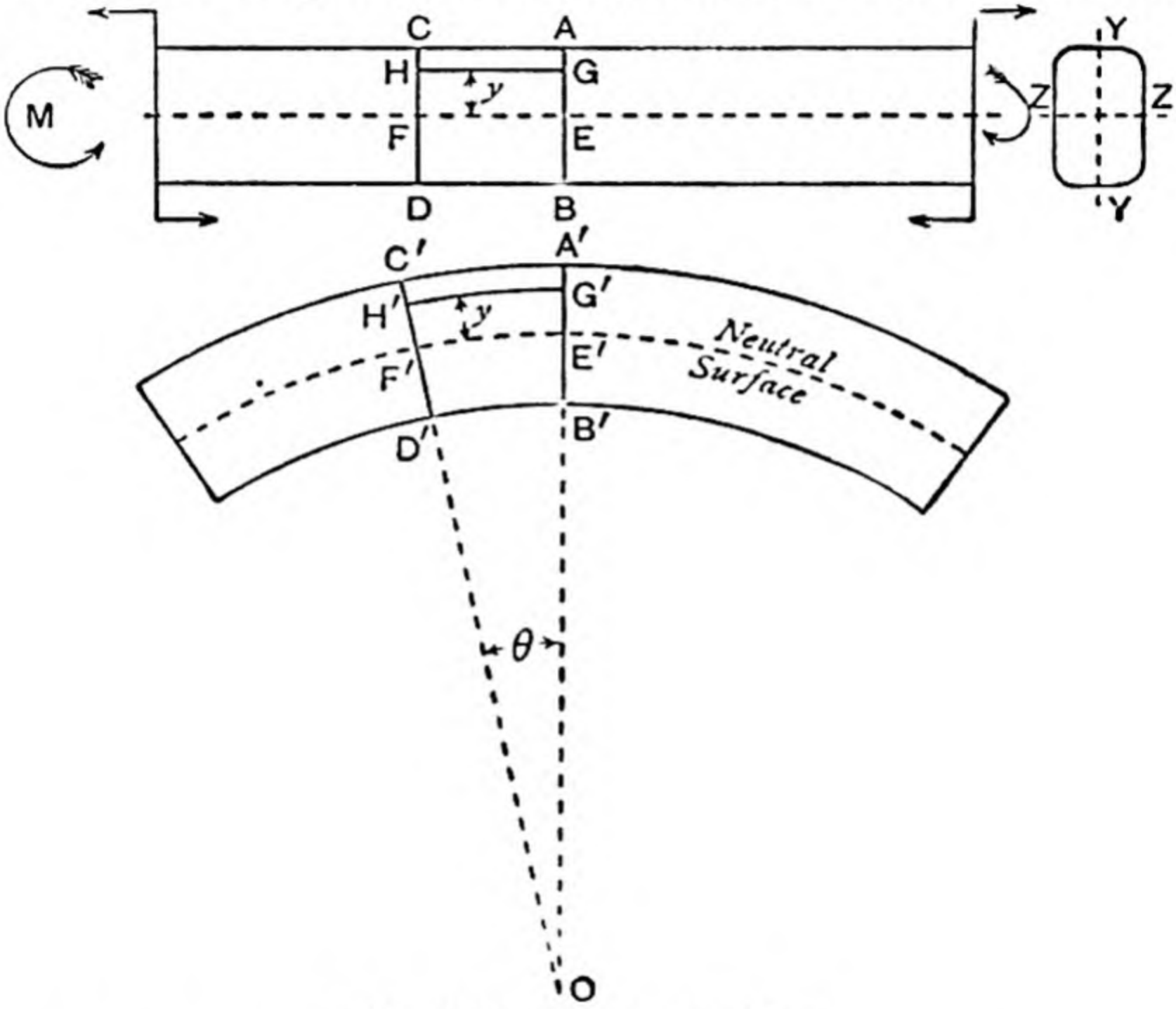


FIG. 157.—Simple bending.

surface. Its line of intersection ZZ with a transverse section is called the *neutral axis* of that section.

Suppose the section A′B′ and C′D′ produced to intersect at an angle θ (radians), in a line perpendicular to the figure and represented by O, and that the radius of curvature OE′ of the neutral surface E′F′ about O is R. Let y be the height (E′G′) of any layer (H′G′) of material originally parallel to the neutral surface FE. Then

$$\frac{H'G'}{E'F'} = \frac{(R+y)\theta}{R\theta} = \frac{R+y}{R}$$

and the proportional strain at the layer H′G′ is

$$e=\frac{\text{H′G′}-\text{HG}}{\text{HG}}=\frac{\text{H′G′}-\text{E′F′}}{\text{E′F′}}=\frac{(\text{R}+y)\theta-\text{R}\theta}{\text{R}\theta}=\frac{y}{\text{R}}$$

The longitudinal tensile-stress intensity p at a height y from the neutral surface, provided the limit of elasticity has not been exceeded, is therefore

$$p=\text{E}\,.\,e=\text{E}\,.\,\frac{y}{\text{R}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where E is Young's modulus of elasticity, provided that the layers of material behave under longitudinal stress as if free and are not hindered by the surrounding material, which has not the same intensity of stress. The intensity

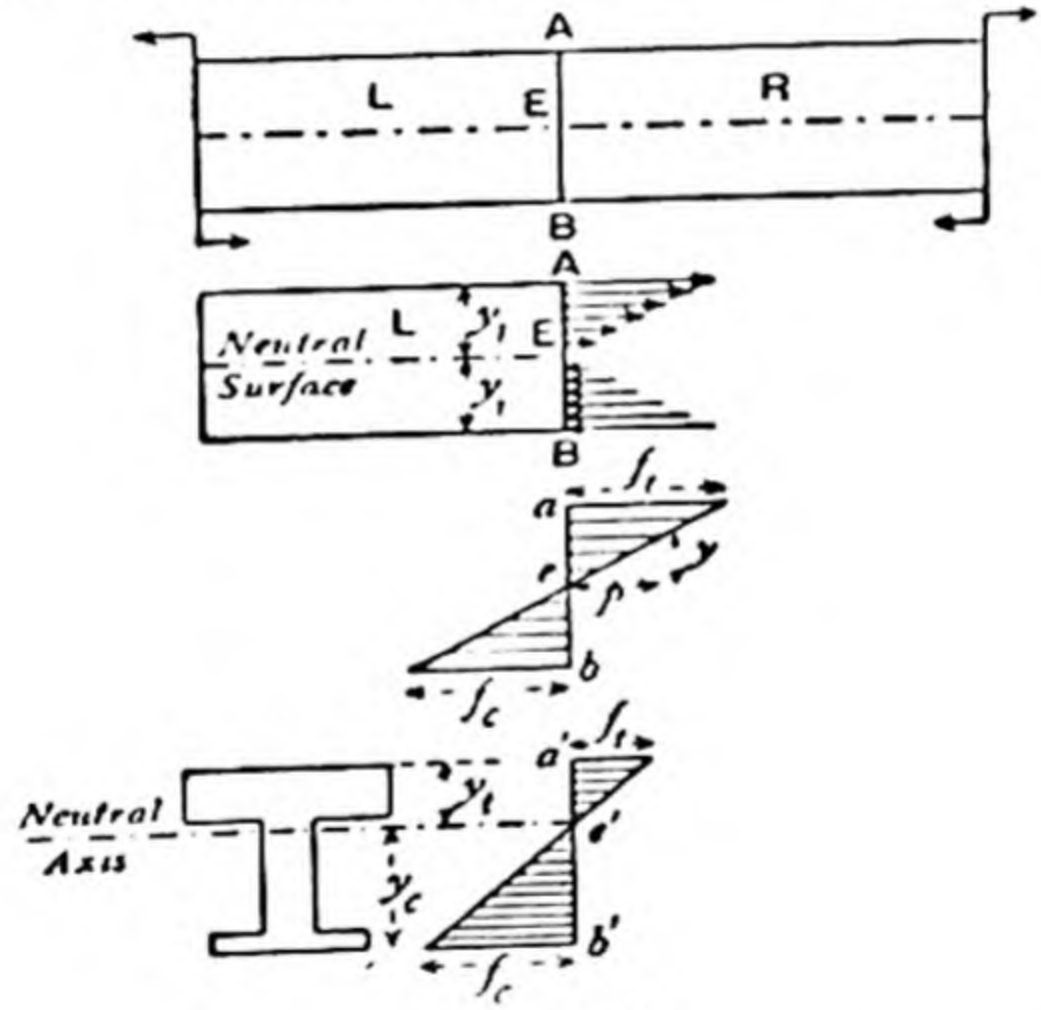


FIG. 158.

of compressive stress will be the same at an equal distance y on the opposite side of the neutral surface, provided E is the same in compression as in tension.

The intensity of direct longitudinal stress p at every point in the cross-section is then proportional to its distance from the neutral axis ; its value at unit distance (*i.e.* at $y=1$) is E/R, and it reaches its greatest value at the boundary furthest from the neutral surface. The variation in intensity of longitudinal stress is as shown in Fig. 158,

where the arrow-heads denote the direction of the force exerted by the portion R on the portion L at the section AB. Since the stresses on opposite sides of the neutral surface are of opposite sign or kind, they may be represented as at *aeb*.

114. Position of the Neutral Axis. The beam has been supposed subjected to pure couples only, and therefore the portion, say, to the left of the section AB (Figs. 157 and 158), being in equilibrium under one externally applied couple and the forces acting across AB, these latter forces must exert a couple balancing the external one in the plane of bending. The (vertical) shearing force being *nil*, the internal forces exerted across AB are wholly horizontal (or longitudinal), and since they form a couple, the total tensile forces must balance the compressive ones, *i.e.* the algebraic sum of the horizontal internal forces must, like the external ones, be zero. Putting this statement in symbols, we can find the position of the neutral axis. The cross-section of the beam in Fig. 157 is symmetrical about a horizontal axis, but this is not necessary to the argument. Taking any other forms of cross-sections symmetrical about the plane of bending YY, as in Fig. 159, let δa or $z \,.\, \delta y$ be an elementary strip of its area parallel to the neutral axis ZZ, z being the (variable) width of the section. Then, the total horizontal force being zero

$$\Sigma(p \,.\, \delta a)=0 \quad \text{or} \quad \Sigma(p \,.\, z \,.\, \delta y)=0$$

and since by (1), Art. 113

$$p=\frac{E}{R} \,.\, y$$

$$\frac{E}{R}\Sigma(y \,.\, \delta a)=0 \quad \text{or} \quad \frac{E}{R}\Sigma(y \,.\, z\delta y)=0 \quad . \quad . \quad (2)$$

the quantity $\Sigma(y \,.\, \delta a)$ or $\Sigma(y \,.\, z \,.\, \delta y)$ represents the total moment of the area of section about the neutral axis, and this can only be zero if the axis passes through the centre of gravity or centroid of the section.

The use of the value $\frac{E}{R} \,.\, y$ for p in all parts of the cross-section involves the assumption that the value of E is

the same in compression as in tension, an assumption justified by experiment within the limits of elasticity.

Assumptions made in the Theory of Simple Bending. It may be well to recall the assumptions made in the above theory of "simple bending" under the conditions stated—

(1) That plane transverse sections remain plane and normal after bending.

(2) That the material is homogeneous, isotropic, and obeys Hooke's law, and the limits of elasticity are not exceeded.

(3) That every layer of material is free to expand or contract longitudinally and laterally under stress, as if separate from other layers. Otherwise, E in the relation (1), Art. 13, would not be Young's modulus, but some

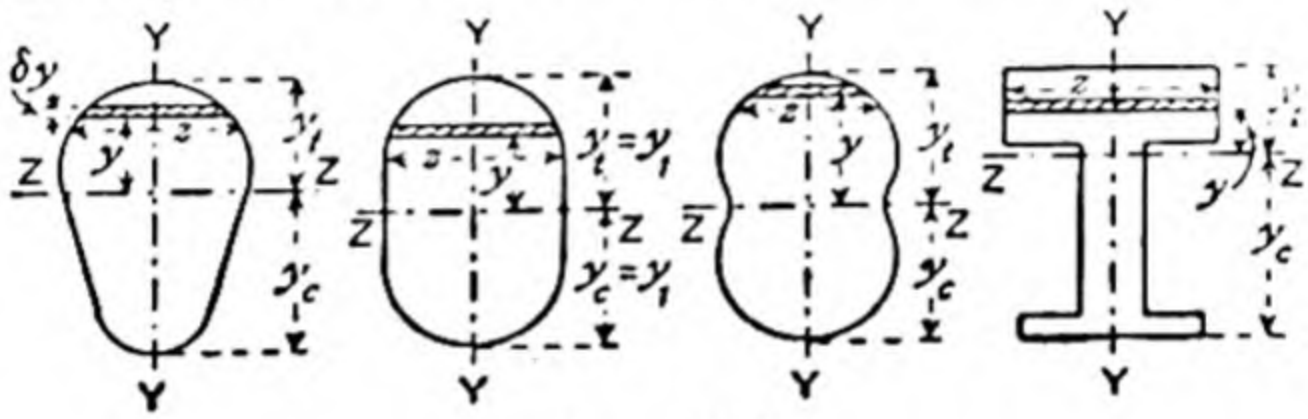


FIG. 159.

modified elastic constant; but the relation would otherwise remain unaltered.

(4) That the modulus of direct elasticity (Young's) has the same value in compression as for tensile strains.

115. Value of the Moment of Resistance. Having found the intensity of longitudinal stress ($p = E \cdot y/R$) at any distance y from the neutral axis, and knowing that these longitudinal internal forces form a couple equal to the bending moment at every section, it remains to express the value of the couple, which is called the moment of resistance (see Art. 110), in terms of the dimensions of the cross-section, and the intensity of stress produced.

Using Fig. 159, as in the previous article, the elementary area of cross-section, at a distance y from the neutral axis is δa, or $z \cdot \delta y$, and the intensity of stress upon it is

$$p = \frac{E}{R} y$$

The total stress on the elementary area is

$$p \,.\, \delta a \quad \text{or} \quad p \,.\, z \,.\, \delta y$$

and the moment of this stress is

$$p \,.\, y \,.\, \delta a \quad \text{or} \quad p \,.\, z \,.\, y \,.\, \delta y$$

and the total moment throughout the section is

$$\mathrm{M} = \Sigma(p \,.\, y \,.\, \delta a) \quad \text{or} \quad \mathrm{M} = \Sigma(p \,.\, z \,.\, y \,.\, \delta y)$$

and putting

$$p = \frac{\mathrm{E}}{\mathrm{R}} \,.\, y \text{ (Art. 113)}$$

$$\mathrm{M} = \frac{\mathrm{E}}{\mathrm{R}} \Sigma(y^2 \,.\, \delta a) \quad \text{or} \quad \frac{\mathrm{E}}{\mathrm{R}} \Sigma(z y^2 \delta y) \quad . \quad . \quad (3)$$

The sum $\Sigma(y^2 \delta a)$, or $\Sigma(z y^2 \delta y)$, represents the limiting value of the sum of the products of elements of area, multiplied by the squares of their distances from the axis, when the elements of area are diminished indefinitely, and is usually called the Moment of Inertia or Second Moment of the area of the section about the axis. The values of the moments of inertia for various sections are dealt with in Art. 117 and Appendix. If we denote the moment of inertia of the area of the section by I, so that

$$\Sigma(y^2 \delta a) = \Sigma(z y^2 \delta y) = \mathrm{I}$$

the formula (3) becomes

$$\mathrm{M} = \frac{\mathrm{E}}{\mathrm{R}} \mathrm{I} \quad \text{or} \quad \frac{\mathrm{M}}{\mathrm{I}} = \frac{\mathrm{E}}{\mathrm{R}} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and since by (1), Art. 113, $\frac{\mathrm{E}}{\mathrm{R}} = \frac{p}{y}$ (the stress intensity at unit distance from the neutral axis), we have

$$\frac{p}{y} = \frac{\mathrm{M}}{\mathrm{I}} = \frac{\mathrm{E}}{\mathrm{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

These relations are important and should be remembered. If we put this relation in the form

$$p = \frac{\mathrm{M}}{\mathrm{I}} \,.\, y \quad \text{or} \quad \frac{\mathrm{E}}{\mathrm{R}} \,.\, y$$

we have the intensity of longitudinal stress at a distance y from the neutral axis, in terms of the bending moment and dimensions (I) of cross-section, or in terms of the radius of curvature and an elastic constant for the material. The extreme values of p, tensile and compressive, occur at the layers of material most remote from the neutral axis. Thus, in Figs. 158 and 159, if the extreme layers on the tension and compression sides are denoted by y_t and y_c respectively, f_t and f_c being the extreme intensities of tensile and compressive stress respectively,

$$\frac{p}{y}=\frac{f_t}{y_t}=\frac{f_c}{y_c}=\frac{M}{I}=\frac{E}{R}$$

or,
$$f_t=M\cdot\frac{y_t}{I} \qquad f_c=M\cdot\frac{y_c}{I}$$

or,
$$M=f_t\cdot\frac{I}{y_t}=f_c\cdot\frac{I}{y_c} \qquad \ldots \quad (6)$$

The variation of intensity of stress for an unsymmetrical section is shown in Fig. 158 at $a'e'b'$.

Modulus of Section. For sections which are symmetrical about the neutral axis, the distances y_t and y_c will be equal, being each half the depth of the section. If we denote the half depth by y_1, and the equal intensities of extreme or skin stress by f_1, so that

$$M=f_1\frac{I}{y_1}$$

the quantity $\frac{I}{y_1}$ is called the modulus of section, and is usually denoted by the letter Z, so that

$$M=f_1Z \quad \text{or} \quad f_1=M/Z \qquad \ldots \quad (7)$$

the moment of resistance (M) being proportional to the greatest intensity of stress reached and to the modulus of section.

In the less usual case of unsymmetrical sections, the modulus of section would have the two values

$$\frac{I}{y_t} \quad \text{and} \quad \frac{I}{y_c}$$

which may be denoted by Z_t and Z_c, so that the relation (6) becomes

$$M = f_t Z_t = f_c Z_c \quad . \quad . \quad . \quad . \quad . \quad (8)$$

EXAMPLE 1. To what radius of curvature may a steel beam of symmetrical section, 12 inches deep, be bent without the skin stress exceeding 5 tons per square inch? (E = 13,500 tons per square inch.)

Since $$\frac{E}{R} = \frac{f_1}{y_1} \qquad \therefore R = \frac{E y_1}{f_1}$$

y_1 being the half depth, which is 6 inches.

Hence

$$R = \frac{13{,}500 \text{ tons per sq. in.} \times 6 \text{ in.}}{5 \text{ tons per sq. in.}} = 16{,}200 \text{ in. or } 1350 \text{ feet.}$$

EXAMPLE 2. If the elastic limit is not exceeded, find the stress induced in a strip of spring steel, $\frac{1}{20}$ inch thick, by bending it round a drum 2·5 feet diameter? (E = 13,500 tons per square inch.)

$$f_1 = \frac{E y_1}{R}$$

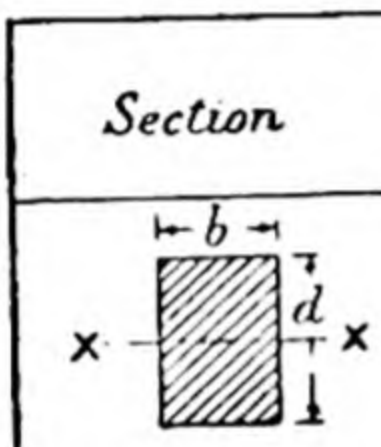
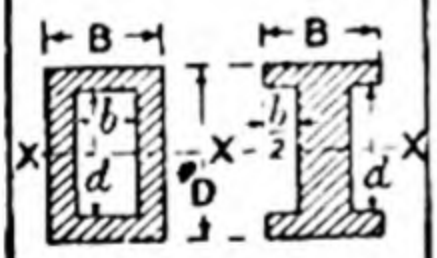
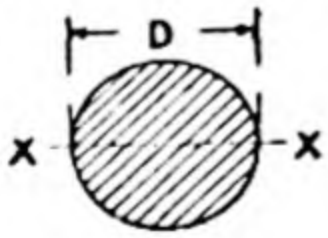
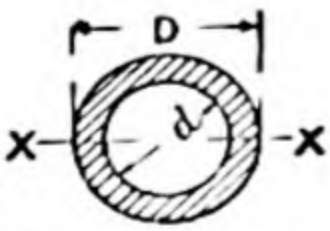
The greatest value of y is $\frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$ inch. The radius being 15 inches

$$f_1 = \frac{13{,}500 \text{ tons per sq. in.} \times \frac{1}{40} \text{ in.}}{15 \text{ in.}} = 22{\cdot}5 \text{ tons per sq. in.}$$

116. Ordinary Bending. The case of *simple bending*, dealt with in the previous articles, refers only to bending where shearing force is absent, but such instances are not usual, and generally bending action is accompanied by shearing force, which produces a (vertical) shear stress across transverse sections of the beam (see Figs. 147 to 154, etc.). In such cases the forces across any section at which the shearing force is not zero have not only to balance a couple, but also the shearing force at the section, and, therefore, at points in the cross-section there will be tangential as well as normal longitudinal stresses. These, however, are relatively small in beams which are long in comparison with their cross-sectional dimensions.

For most practical cases the foregoing theory of "Simple Bending" (Arts. 113, 114 and 115) is quite sufficient, and gives results which enable the engineer to design beams and structures, and calculate their stresses and strains with a considerable degree of approximation. It may be noticed that in many cases of continuous loading the *greatest* bending moment occurs as a mathematical maximum at the sections for which the shearing force is zero (Art. 111, and Figs. 150, 151, and 152), and for which the conditions correspond with those for *simple* flexure.

117. Calculation of Bending Stress Intensities. The numerical calculation of bending stress intensity in a beam of given dimensions due to a bending moment M is now

Section	*Moment of inertia* I_{XX}.	*Modulus of section* (Z).
	$\frac{1}{12}bd^3$	$\frac{1}{6}bd^2$
	$\frac{1}{12}(BD^3-bd^3)$	$\frac{1}{6}\left(\frac{BD^3-bd^3}{D}\right)$
	$\frac{\pi}{64}D^4$	$\frac{\pi}{32}D^3$
	$\frac{\pi}{64}(D^4-d^4)$	$\frac{\pi}{32}\left(\frac{D^4-d^4}{D}\right)$

merely a matter of substitution in the expression given in Art. 115, viz. at a distance y from the neutral axis the stress intensity is

$$p = My/I \quad \text{or} \quad M/Z \times y/y_1 \quad . \quad . \quad . \quad (1)$$

and at the outside edge of the section, *i.e.* furthest from the neutral axis of a section symmetrical about the neutral axis, p becomes

$$f_1 = My_1/I \quad \text{or} \quad f_1 = M/Z \quad . \quad . \quad . \quad . \quad (2)$$

where Z is the modulus of section.

The value of the moment of inertia (I), or second moment, of area of cross-section, and of the modulus of section Z is shown for each of five common forms of section in the table on preceding page.

The expressions for I are readily found by means of the integral calculus [1] and the values of Z, by dividing that of I by half the depth, *i.e.* by $\frac{1}{2}d$ in the case of the rectangle and by $\frac{1}{2}$D in the case of the circle. Caution is needed in the case of the hollow section ; the I is the difference of the I's of the two areas but the Z is not the difference of the Z's of the two areas but the difference of the I's divided by the full depth of the section, *e.g.* $\frac{1}{6}(BD^3 - bd^3)/D$ and *not* $\frac{1}{6}(BD^2 - bd^2)$. Hollow sections are economical because they have much of their material where stress intensity is high, namely, far from the neutral axis.

The value of Z for a section which is not geometrically simple can be found in various ways. $Z = \Sigma(y^2\delta a)/y_1$ may be written $\Sigma(y\delta a \times y/y_1)$, which is the total moment of an area ; not the area of section but one in which every element of area δa is reduced in the ratio (y/y_1) of its distance from the neutral axis to that of the distance of the

[1] For a rectangular section

$$I = \int_{-d/2}^{d/2} by^2 dy = \left[by^3/3\right]_{-d/2}^{d/2} = bd^3/12$$

or $\qquad k_G^2 = d^2/12 \qquad Z = I/\frac{1}{2}d = bd^2/6$

For a circular area, about a axis perpendicular to the figure, take x as the radius of an element of width dx

$$I_0 = \int_0^{D/2} x^2 \times 2\pi x dx = \left[2\pi x^4/4\right]_0^{D/2} = \pi D^4/32$$

and using theorem (2) of the appendix, this is equal to the sum of I_X and I_Y, the two equal moments of inertia about diameters, hence,

$$I_X = I_Y = \tfrac{1}{2}I_0 = \pi D^4/64 \text{ (or see (9) of appendix)}$$

and $\qquad Z = I_X/\frac{1}{2}D = \pi D^3/32$

most distant part of the section from the neutral axis. Such reduction can be accomplished graphically or arithmetically.

EXAMPLE 1. A timber beam of rectangular section is to be simply supported at the ends and carry a load of $1\frac{1}{2}$ ton at the middle of a 16-feet span. If the maximum stress is not to exceed $\frac{3}{4}$ ton per square inch and the depth is to be twice the breadth, determine suitable dimensions.

The reactions at the ends are each $\frac{3}{4}$ ton, and the bending moment at the centre is

$$\tfrac{3}{4} \text{ ton} \times 8 \times 12 \text{ inches} = 72 \text{ ton-inches}$$

The modulus of section (Z) is given by

$\frac{3}{4}$ ton per sq. in. $\times Z = 72$ ton-inches, $\therefore Z = 96$ (inches)^3;

and if $b = \frac{1}{2}d$

$$\tfrac{1}{6}bd^2 = \tfrac{1}{12}d^3 = 96 \text{ (in.)}^3$$

$$d = \sqrt[3]{1152} = 10{\cdot}5 \text{ inches nearly}$$

$$b = 5{\cdot}25 \text{ inches}$$

EXAMPLE 2. The moment of inertia of a symmetrical section being 2654(inch)^4 units, and its depth 24 inches, find the longest span over which, when simply supported, a beam could carry a uniformly distributed load of 1·2 ton per foot run, without the stress exceeding 7·5 tons per square inch.

If l = span, the load per inch run being 1·2/12, or 0·1 ton, the maximum bending moment which occurs at mid-span is

$$\text{M} = \tfrac{1}{8} \times 0{\cdot}1 \text{ ton per inch} \times l^2 \text{ (inches)}^2 = l^2/80 \text{ ton-inches.}$$

And since $\text{M} = f_1 \cdot \dfrac{\text{I}}{y_1}$, and y_1 = half depth = 12 inches,

$$l^2/80 \text{ ton inch} = 7{\cdot}5 \text{ tons per sq. in.} \times \tfrac{2654}{12} \text{ (inches)}^3$$

$$l^2 = 80 \times 7{\cdot}5 \times 2654/12 \text{ (inches)}^2$$

$$= 132{,}700 \text{ (inches)}^2$$

$$l = 364 \text{ inches or } 30 \text{ ft. } 4 \text{ in.}$$

EXAMPLE 3. A steel I-section girder is 10 inches deep and 6 inches wide, the flanges being 0·75 in. thick and web 0·4 in. If this beam is simply supported at its ends over a span of 16 feet, what central load could it carry without the bending stress exceeding 7·5 tons per sq. in.?

Using the values of I given in the table for this section and its equivalent, the hollow rectangle,

$$D = 10 \text{ in.}, \quad B = 6 \text{ in.}, \quad d = 10 \text{ in.} - 2 \times 0{\cdot}75 \text{ in.} = 8{\cdot}5 \text{ in.}$$

$$b = 6 \text{ in.} - 0{\cdot}4 \text{ in.} = 5{\cdot}6 \text{ in.}$$

$$I = (6 \times 10^3 - 5{\cdot}6 \times 8{\cdot}5^3)/12 = 2561/12 = 213{\cdot}4 (\text{in.})^4$$

$$Z = 213{\cdot}4/5 = 42{\cdot}68 (\text{in.})^3$$

With a maximum bending stress of 7·5 tons per sq. in. this will give a moment of resistance of 7·5 ton/(in.)^2 × 42·68 (in.)^3 = 320·1 ton-inches. The greatest B.M. to be resisted will be midway between the ends. If W is the central load, the end reactions or supporting forces will be W/2 and the B.M. at the centre using inch units will be

$$\tfrac{1}{2}W \times 8 \times 12 = 48W$$

Hence $W \times 48 \text{ in.} = 320{\cdot}1$ ton-inches

$$W = 320{\cdot}1 \text{ ton-in.}/48 \text{ in.} = 6{\cdot}67 \text{ tons.}$$

EXAMPLE 4. What size of (solid) rectangular section with depth twice the breadth would be required, with the same maximum intensity of stress, for the load in Example 3? And how much more steel would be required?

$$Z = bd^2/6 \text{ and if } b = d/2, \; Z = d^3/12$$

and this must be the same as for the I section, viz.

$$d^3/12 = 42{\cdot}68 (\text{in.})^3 \text{ or } d^3 = 12 \times 42{\cdot}68 = 512 (\text{in.})^3$$

$$d = \sqrt[3]{512} = 8 \text{ inches}; \quad \therefore b = 4 \text{ in.}$$

Area of section $= 8 \times 4 = 32$ sq. in.

Area of I section $= 2 \times 6 \times 0{\cdot}75 + 8{\cdot}5 \times 0{\cdot}4 = 12{\cdot}4$ sq. in.

Ratio of solid to hollow section $= 32/12{\cdot}4 = 2{\cdot}58$, *i.e.* the solid section is 1·58 or 158 per cent. greater and the weight of steel used is 2·58 times as great. This illustrates the

economy of using an I-section beam. For reasons of manufacture, the flanges actually are not rectangular in section but are tapered and have rounded corners ; otherwise the section chosen is representative of a rolled girder section.

118. Twisting. Stress and Strain in Torsion. When a cylindrical bar is twisted by equal and opposite couples having a common axis which coincides with that of the bar, it is subjected to pure torsion. The stress at any point in a cross-section is one of pure shear and the strain is such that any section perpendicular to the axis of the bar makes a small rotatory movement relative to other similar sections. The nature of the strain within the elastic limit is illustrated in Fig. 160, which represents a solid cylindrical bar

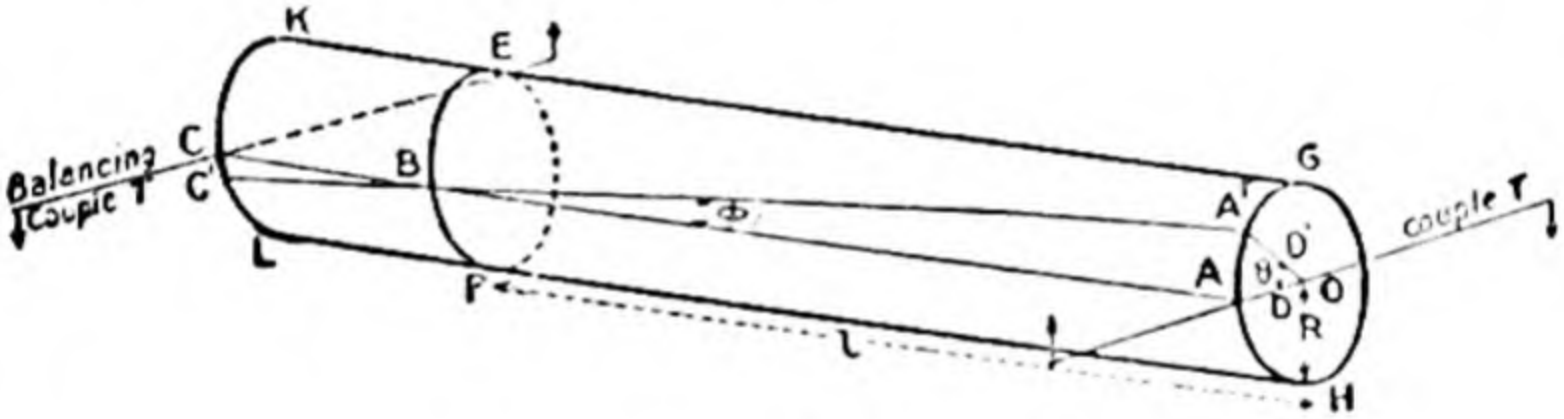


FIG. 160.

in equilibrium under two equal and opposite couples at its ends. A line ABC on the curved surface, originally straight and parallel to the axis of the bar, after the strain takes place becomes part of a helix A'BC', which everywhere makes an angle ϕ_1 with lines such as AB, which are parallel to the axis ; the constancy of this angle would be apparent if the curved surface were developed into a plane one, when A'BC' would be a straight line. The angle ϕ_1 is the shear strain (Art. 94) for all the material at the curved surface, and the elastic strain being small,

$$\phi_1 = \frac{AA'}{AB} = \frac{f_s}{N} \text{ (radians)} \quad . \quad . \quad . \quad . \quad (1)$$

where f_s is the intensity of shear stress (q) at the surface, and N is the modulus of rigidity (see Art. 97).

For any point such as D, distant r from the centre of the cross-section, the shear strain ϕ and intensity of shear stress q are similarly connected by the equation

$$\phi = \frac{DD'}{AB} = \frac{q}{N} \text{ (radians)} \quad . \quad . \quad . \quad . \quad (2)$$

The radial line originally at OA, after straining occupies a position A'O, the angle of twist AOA' being θ in a length AB or l. If the radius of the bar is R, from (1)

$$\theta = \frac{AA'}{AO} = \frac{l\phi_1}{R} \text{ (radians)} \quad . \quad . \quad . \quad (3)$$

and similarly from (2)

$$\theta = \frac{DD'}{OD} = \frac{l\phi}{r} \text{ (radians)} \quad . \quad . \quad . \quad . \quad (4)$$

From (1) and (3)

$$f_s = \phi_1 N = \frac{R\theta}{l} . N \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and from (2) and (4)

$$q = \phi N = \frac{r\theta}{l} N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

the intensity of shear stress on the cross-section being at every point proportional to the distance r from the axis, varying from zero at the axis to the extreme value f_s at the circumference.

119. Relation between Twisting Effort, Torsional Strain and Stress. The relation between a given torsional straining action and the effects produced within the elastic limit on a cylindrical bar of given dimensions may be calculated from the principles of equilibrium and the formulæ of the previous article. Considering the equilibrium of the piece EGHF of the circular bar, Fig. 160, the only external forces upon it are those of the couple T at the end AGH, and those exerted by the piece KEFL in the shear stress across the plane EBF; hence the latter must reduce to a couple of magnitude T, and of opposite sense to that

applied at the end AGH. If Fig. 161 represents the ring, force on an elementary ring of radius r and width δr is

$$q \times 2\pi r \delta r$$

and from (5) and (6), Art. 118

$$q = f_s \,.\, r/\mathrm{R}$$

hence $$q \times 2\pi r \delta r = f_s \,.\, 2\pi r^2 \delta r/\mathrm{R}$$

and the moment of this about the axis is

$$\delta \mathrm{T} = 2\pi f_s r^3 \delta r/\mathrm{R}$$

Dividing the whole section into elementary concentric

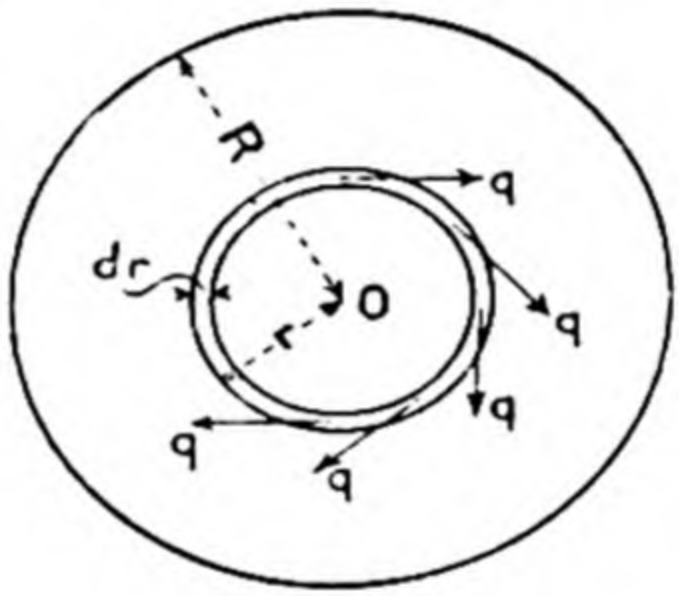


FIG. 161.

rings and summing the moments, the total couple exerted across the section is

$$\mathrm{T} = 2\pi \frac{f_s}{\mathrm{R}} \int_0^{\mathrm{R}} r^3 dr = \frac{\pi}{2} f_s \mathrm{R}^3 \quad \text{or} \quad \frac{\pi}{16} \cdot f_s \,.\, \mathrm{D}^3 \,. \quad . \quad (1)$$

where D or 2R is the diameter of the bar. The quantity

$$2\pi \int_0^{\mathrm{R}} r^3 dr = \pi \mathrm{R}^4/2 \quad \text{or} \quad \pi \mathrm{D}^4/32 = \mathrm{J} \text{ (say)} \quad (1\text{A})$$

is the polar moment of inertia of the area of cross-section about the axis (written I_0 in the footnote to Art. 117), and (1) may be written

$$\mathrm{T} = f_s \mathrm{J}/\mathrm{R} = q\mathrm{J}/r \quad \text{or} \quad f_s = \mathrm{TR}/\mathrm{J} \quad \text{and} \quad q = \mathrm{T}r/\mathrm{J} \,. \quad (2)$$

It should be remembered that if, say, inch units of length

are used for the dimensions and for f_s, the same units must be used for T (pound-inches or ton-inches).

In this form (2), the close analogy between the relations connecting the couple, stress, and dimensions for torsion and those for bending (Art. 115) is apparent, and the quantity J/R may be called the polar or torsional modulus of a solid circular section.

From (5), Art. 118

$$\theta = \frac{f_s \cdot l}{R \cdot N} = \frac{T \cdot l}{N \cdot J} \quad \text{or} \quad \frac{32Tl}{\pi D^4 N} \text{ (radians)} \quad . \quad . \quad (3)$$

or T/NJ radians per unit length ; also in degrees

$$\theta = \frac{583Tl}{ND^4} \text{ degrees} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the amount of twist being proportional to the length, and inversely proportional to the (polar) moment of inertia (J) of cross-section about the axis, *i.e.* in a shaft of solid circular section, inversely proportional to the fourth power of the diameter. The product NJ to which the amount of twist is inversely proportional, may be called the torsional rigidity of the shaft.

EXAMPLE 1. What torque must be exerted on (and by) a shaft 2 inches diameter if the maximum shear stress produced is 5000 lb. per sq. in. ?

From (1), $T = (\pi/16) \times 5000$ lb. per sq. in. $\times 2^3$(inches)^3 $= 7{,}854$ lb.-in.

EXAMPLE 2. What maximum shear stress will there be in a steel shaft which is twisted through an angle of one degree in a length of 20 times the diameter of the shaft if N =12,000,000 lb. per sq. in. ?

From (3) since θ is $\pi/180$ radians and $l/2R = 20$ or $l/R = 40$,

$$f_s = \frac{\pi}{180} \times \frac{1}{40} \times 12{,}000{,}000 \text{ lb./(in.)}^2 = 5236 \text{ lb. per sq. in.}$$

Note that this is independent of the diameter of the shaft if twist per 20 diameters is specified.

120. Shaft Diameters for Power Transmission. In the transmission of power through a shaft, the product of the mean twisting moment or torque multiplied by the angle turned through in (radians) gives the work transmitted. Hence, if T is the mean twisting moment in pound-inches, caused in transmitting horse-power (h.p.) at n revolutions per minute

$$T = \frac{12 \times 33{,}000 \times \text{h.p.}}{2\pi n} \quad . \quad . \quad . \quad . \quad (1)$$

The maximum twisting moment will generally be considerably in excess of this amount, as the twisting moment usually varies considerably in driving of all kinds. If some ratio of the maximum to the mean torque be adopted, the maximum twisting moment will be

$$T = \frac{\text{h.p.}}{n} \times \text{coefficient} \quad . \quad . \quad . \quad . \quad (2)$$

and for a shaft transmitting *torsion only* (without bending stress), if f_s be the intensity of safe maximum shear stress, from (1), Art. 109

$$D = \sqrt[3]{\frac{16T}{\pi f_s}} = \sqrt[3]{\frac{\text{h.p.}}{n}} \times \text{numerical coefficient} . \quad (3)$$

a common value of this coefficient for steel shafts being about 3·3. When the maximum twisting moment has been estimated, common working values of f_s are 8000 to 10,000 lb. per square inch. Suitable values of f_s or of the constant in (3) will be found in manuals of machine design. In long shafts the condition that the twist in a given length shall be within some assigned limit may require a larger diameter than considerations of maximum intensity of shear stress.

EXAMPLE 1. Find the maximum intensity of torsional shear stress in a shaft 3 inches diameter transmitting 50 h.p. at 80 revolutions per minute if the maximum twisting moment exceeds the mean by 40 per cent. What is the greatest twist in degrees per foot of length if N $=12 \times 10^6$ lb. per square inch ?

Using inch units,

Mean twisting moment

$$=\text{work per minute/radians per minute}$$
$$=50\times33{,}000\times12 \text{ inch-lb.}/(80\times2\pi) \text{ radians}$$

Maximum twisting moment

$$T=1{\cdot}40\times50\times33{,}000\times12/(80\times2\pi) \text{ lb.-in.}$$
$$=55{,}125 \text{ lb.-in.}$$

Maximum intensity of shear stress

$$T\div(\pi\,)^3/16)$$
$$=55{,}125\times16/27\pi \text{ lb. per sq. in.}$$
$$=10{,}400 \text{ lb. per sq. in.}$$

Maximum twist per 12 inch length

$$=32\times55{,}125\times12/(\pi\times3^4\times12\times10^6) \text{ radian}$$

Maximum twist in degrees per ft.

$$=\frac{32\times55{,}125\times12}{\pi\times81\times12\times10^6}\times\frac{180}{\pi}=0{\cdot}398 \text{ degree}$$

EXAMPLE 2. What h.p. is transmitted by a shaft 3 inches in diameter at 80 revolutions per minute if the twist is constant at 20° in a length of 50 feet if the modulus of rigidity is 12,000,000 lb. per sq. in. ?

Twist in radians on 50 ft. length (θ)

$$=20\times\pi/180=0{\cdot}349 \text{ radian}$$

Peripheral displacement at surface

$$=1{\cdot}5\times0{\cdot}349 \text{ inch.}$$

Shear strain (ϕ) on length 50×12 inches (see Fig. 160)

$$=1{\cdot}5\times0{\cdot}349/600=0{\cdot}0008725 \text{ radian}$$

Shear stress at shaft surface (f_s)

$$=0{\cdot}0008725\times12{,}000{,}000$$
$$=10{,}270 \text{ lb. per sq. in.}$$

Torque T (in lb.-in.)

$$=10{,}270 \text{ lb./sq. in} \times \pi \times 3^3/16 \text{ (in.)}^3$$
$$=10{,}270 \times \pi \times 27/16 = 54{,}460 \text{ lb.-in.}$$
$$\text{h.p.} = 54{,}460 \times 2\pi \times 80/(33{,}000 \times 12) = 69{\cdot}13 \text{ h.p.}$$

121. Hollow Circular Shafts. The intensity of stress in a circular shaft being, for all points in a cross-section, proportional to the distance from the axis, when the material at the outside of a solid shaft reaches the maximum safe limit of stress, that about the centre is only carrying a much smaller stress. In the case of a hollow shaft the stress intensity is, as before, everywhere proportional to the distance from the axis, but it varies from a maximum to some smaller value, but not to zero. With the same magnitude of maximum stress the average intensity of stress is greater, and consequently for a given cross-sectional area a greater torque can be resisted.

Let R_1 or $D_1/2$ and R_2 or $D_2/2$ be the external and internal radii respectively of a hollow shaft; then Art. 119 (1), integrated between limits, gives—

$$T = 2\pi \frac{f_s}{R_1} \int_{R_2}^{R_1} r^3 dr = \frac{\pi}{2} f_s \frac{R_1^4 - R_2^4}{R_1} = \frac{\pi}{16} f_s \cdot \frac{D_1^4 - D_2^4}{D_1} . \quad (1)$$

the value of J in (2) and (3), Art. 119, being—

$$J = \frac{\pi(R_1^4 - R_2^4)}{2}$$

for a hollow shaft. The angle of twist

$$\theta = \frac{32Tl}{\pi(D_1^4 - D_2^4)N} \text{ (radians)} = \frac{583Tl}{N(D_1^4 - D_2^4)} \text{ degrees} . \quad (2)$$

Comparing the strength (or twisting resistance for a given extreme intensity of stress) per unit area of cross-section or per unit weight of a hollow shaft with that of a solid shaft, both having the same external diameter

$$\frac{\text{hollow}}{\text{solid}} = \frac{R_1^4 - R_2^4}{R_1(R_1^2 - R_2^2)} \div \frac{R_1^3}{R_1^2} = 1 + \left(\frac{R_2}{R_1}\right)^2$$

which tends to the limiting ratio 2 as R_2 approaches R_1,

i.e. in a thin tube. The ratio of the torsion rigidities of the two shafts is the same as that of their strengths.

EXAMPLE 1. What horse-power could be transmitted by a hollow steel shaft 12 inches external and 8 inches internal diameter at 120 revolutions per minute if the maximum torque exceeds the mean by 30 per cent. and the maximum shear stress is not to exceed 8000 lb. per sq. inch? What would be the maximum angle of twist on a length of 20 feet of the shaft if the modulus of rigidity of the steel is 12,000,000 lb. per sq. inch?

Maximum torque

$$=8000 \text{ lb. per sq. in.} \times \pi(12^4-8^4)/(16\times 12) \text{ (inch)}^3$$

$$=8000\times\pi\times 208\times 80/192 \text{ lb.-in.}$$

$$=2{,}179{,}000 \text{ lb.-in.}$$

Mean torque

$$=2{,}179{,}000/1{\cdot}3=1{,}676{,}000 \text{ lb.-in.}$$

h.p. at 120 r.p.m. $=120\times 2\pi\times 1{,}676{,}000/(12\times 33{,}000)$

$$=3192 \text{ h.p.}$$

Shear strain at outside surface

$$=8000/12{,}000{,}000=1/1500 \text{ radian}$$

Movement of point on surface relative to one 20 feet (axially) away

$$=20\times 12/1500=0{\cdot}16 \text{ inch}$$

Angle of twist (on 6 in. radius)

$$=0{\cdot}16/6 \text{ radians}$$

$$=0{\cdot}16\times 180/(6\times\pi)\text{degrees}$$

$$=1{\cdot}528 \text{ degrees.}$$

Checking on the previous maximum torque, from (2), Art. 121

$$\theta=\frac{583\times 2{,}179{,}000\times 240\times 32}{\pi(12^4-8^4)\times 12{,}000{,}000}=1{\cdot}528 \text{ degrees.}$$

Example 2. Find the diameters for a hollow shaft to transmit 4000 h.p. at 100 revolutions per minute if the maximum stress is not to exceed 8,000 lb. per square inch and the maximum torque exceeds the mean by 40 per cent. Take the internal as 0·6 of the external diameter. How much more steel would be required for a solid shaft with the same limit of stress?

Maximum torque

$=1{\cdot}4 \times$ work per min./angular speed in radians per min.

$=1{\cdot}4 \times 4000 \times 33{,}000 \times 12/200\pi$ lb.-in.

$=3{,}530{,}000$ lb.-in.

Let D_1 be the external diameter; $0{\cdot}6\ D_1$ = internal diameter.

Polar modulus of section

$$=\pi\{D_1^4-(0{\cdot}6D_1)^4\}/16D$$

$$=\pi D_1^3(1-0{\cdot}1296)/16=0{\cdot}1709D_1^3$$

Maximum torque

$=0{\cdot}1709D_1^3 \times 8000$ lb./sq. in. $=1367D_1^3$ lb./sq. in.

Equating this to above maximum torque

$$D_1^3=3{,}530{,}000/1367\ (\text{inch})^3=2582\ (\text{inch})^3$$

External diameter $D_1=\sqrt[3]{2582}=13{\cdot}72$ inches

Internal diameter $=0{\cdot}6D_1=8{\cdot}23$ inches

(say, 13·75 in. and 8·25 in.)

For a solid shaft of diameter D, the torque is

8000 lb. per sq. in. $\times \pi D^3/16=3{,}530{,}000$ lb.-in.

$$D^3=3{,}530{,}000/1571=2247\ (\text{in.})^3$$

$$D=\sqrt[3]{2247}=13{\cdot}1 \text{ in.}$$

The ratio of weight of solid to hollow shaft is that of their cross-sectional areas, viz. $13{\cdot}1^2/(13{\cdot}72^2-8{\cdot}23^2)$ $=171{\cdot}6/120{\cdot}5=1{\cdot}421$, that is, the solid shaft requires 42·1 per cent. more steel than the hollow one.

EXAMPLES XII

1. A cantilever 12 feet long carries loads of 3, 7, 4, and 6 tons at distances 0, 2, 5, and 8 feet respectively from the free end. Find the bending moment and shearing force at the fixed end and at the middle section of the beam.

2. A cantilever 10 feet long weighs 25 lb. per foot run, and carries a load of 200 lb., 3 feet from the free end. Find the bending moment at the support, and draw the diagrams of shearing force and bending moment.

3. A beam rests on supports 16 feet apart, and carries, including its own weight, a load of 2 tons (total) uniformly distributed over its whole length and concentrated loads of $1\frac{1}{2}$ ton and $\frac{1}{2}$ ton, 5 feet and 9 feet respectively from the left support. Find the bending moment 4 feet from the left-hand support, and the position and magnitude of the maximum bending moment.

4. A horizontal beam, AB, 30 feet long is supported at A and at C 20 feet from A, and carries a load of 7 tons at B and one of 10 tons midway between A and C. Draw the diagrams of bending moment. Where is the B.M. zero between A and C?

5. A girder 40 feet long is supported at 8 feet from each end, and carries a load of 1 ton per foot run throughout its length. Find the bending moment at the supports and at mid-span. Sketch the curve of bending moments.

6. A beam 18 feet long rests on two supports 10 feet apart, overhanging the left-hand one by 5 feet. It carries a load of 5 tons at the left-hand end, 7 tons midway between the supports, and 3 tons at the right-hand end. Find the bending moment at the middle section of the beam and at mid-span.

7. If the beam in the previous example carries an additional load of 1 ton per foot run between the supports, find the bending moment at mid-span.

8. Find the greatest intensity of direct stress arising from a bending moment of 90 ton-inches on a symmetrical section 8 inches deep, the moment of inertia being 75 $(\text{inch})^4$ units.

9. Calculate the moment of resistance of a beam section 10 inches deep, the moment of inertia of which is 145 $(\text{inch})^4$ when the maximum intensity of stress reaches 7·5 tons per square inch.

10. What total distributed load may be carried by a simply supported beam over a span of 20 feet, the depth of section being 12 inches, the moment of inertia being 375 $(\text{inch})^4$, and the allowable intensity of stress 7·5 tons per square inch? What load at the centre might be carried with the same maximum stress intensity?

11. A wooden beam of rectangular section 12 inches deep and 8 inches wide has a span of 14 feet, and carries a load of 3 tons at the middle of the span. Find the greatest stress in the material and the radius of curvature at mid-span. $E = 800$ tons per square inch.

12. What should be the width of a joist 9 inches deep if it has to carry a uniformly spread load of 250 lb. per foot run over a span of 12 feet, with a stress not exceeding 1200 lb. per square inch ?

13. A floor has to carry a load of 3 cwt. per square foot. The joists are 12 inches deep by 4½ inches wide, and have a span of 14 feet. How far apart may the centre lines be placed if the bending stress is not to exceed 1000 lb. per square inch ?

14. Over what length of span may a rectangular beam 9 inches deep and 4 inches wide support a load of 250 lb. per foot run without the intensity of bending stress exceeding 1000 lb. per square inch ?

15. A beam of I section 12 inches deep has flanges 6 inches wide and 1 inch thick, and web ⅞ inch thick. Compare its flexural strength with that of a beam of rectangular section of the same weight, the depth being twice the breadth.

16. A rolled steel joist 10 inches deep has flanges 6 inches wide by ¾ inch thick. Find approximately the maximum intensity of stress produced in it by a load of 15 tons uniformly spread over a span of 14 feet.

17. Find the bending moment which may be resisted by a cast-iron pipe 6 inches external and 4½ inches internal diameter when the greatest intensity of stress due to bending is 1500 lb. per square inch.

18. To what intensity of shear stress will the steel of a 3·5 inches diameter shaft be raised if it is twisted through 1° in every 6 feet, if the modulus of rigidity of the steel is 12,000,000 lb. per square inch ?

19. What torque will produce a shear stress of 8000 lb. per square inch in a shaft 3 inches in diameter ? Through what angle would the shaft be twisted in a length of 10 feet if the modulus of rigidity is 12,000,000 lb. per square inch ?

20. A shaft is required to transmit 100 h.p. at 75 revolutions per minute. What diameter must it be in order to limit the shear stress to 8000 lb. per square inch if the maximum torque exceeds the mean by 30 per cent. ? If the modulus of rigidity is 12,000,000 lb. per square inch, what is the maximum angle of twist in a 10 feet length ?

21. Find the maximum shear stress produced in a shaft 3 inches diameter which transmits 80 h.p. at 120 revolutions per minute if the maximum torque is 1·25 times the mean.

22. What intensity of shear stress will there be in a shaft 16 inches external and 8 inches internal diameter subjected to a torque of 4,000,000 ton-inches ? What will be the twist in a length 20 times the external diameter if $N = 12 \times 10^6$ lb. per square inch ?

23. A hollow shaft is to transmit 400 h.p. at 80 revolutions per minute. If the shear stress is not to exceed 8000 lb. per square inch and the internal is 0·6 of the external diameter, find the external and internal diameters assuming that the maximum torque is 1·4 times the mean.

APPENDIX I

Moments of Inertia and Radii of Gyration. Two theorems are valuable in calculating moments of inertia or radii of gyration of solids or of plane figures about different axes.

Theorem (1). The square of the radius of gyration of a solid body about any axis exceeds that about a parallel axis through its centre of gravity by the square of the distance between these two axes. Thus if k_G is the radius of gyration of a body about an axis through its centre of gravity and k is the radius of gyration about any parallel axis distant l from the c.g., then

$$k^2 = k_G^2 + l^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$I = I_G + Ml^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where I is the moment of inertia of a mass M about an axis distant l from the c.g. and I_G is moment of inertia about a parallel axis through the c.g. of the body.

Similarly

$$k^2 = k_G^2 + l^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$I = I_G + Al^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where I, I_G, k and k_G relate to the moments of inertia (or second moments) of an area A about any axis and about an axis in the same plane and distant l from the central one, *i.e.* the axis through the c.g.

Theorem (2). The sum of the squares of the radii of gyration of an area about two mutually perpendicular axes in its plane is equal to the square of the radius of gyration about an axis through the intersection of the other two axes and perpendicular to the plane of the area. Thus if k_X and k_Y are the radii of gyration of an area A, about axes OX and OY respectively, and k_O is its radius of gyration about an axis OO′ through O (Fig. 162) and perpendicular to the plane of the area shown (and to the diagram)

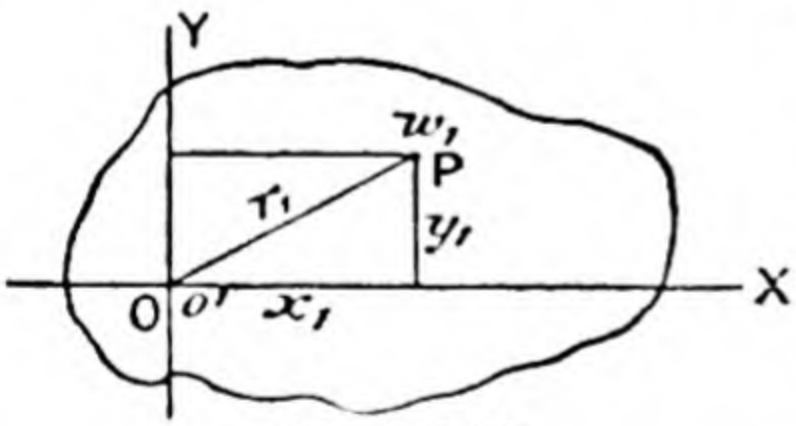


FIG. 162.

Then $k_O^2 = k_X^2 + k_Y^2$. (5)

Also $I_O = I_X + I_Y$. (6)

where $I_O = Ak_O^2$, $I_X = Ak_X^2$ and $I_Y = Ak_Y^2$, these being the moments of inertia (or second moments) of the area A about the axes O, OX, and OY respectively, the axis O being perpendicular to the plane of the area, *i.e.* to Fig. 162. For if δA is an element of area A at P

$$r^2\delta A = x^2\delta A + y^2\delta A \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\Sigma(r^2 . \delta A) = \Sigma(x^2 . \delta A) + \Sigma(y^2 . \delta A) \quad . \quad . \quad . \quad (8)$$

which is equivalent to (6) above or dividing throughout by A gives (5).

Circular Area. From (1A) Art. 119, where $D = 2R$, $I_O = J = \pi D^4/32 = \pi R^4/2$. Hence, about a diameter,

$$I_X = I_Y = J/2 = \pi R^4/4 \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and

$$k^2 = (\pi R^4/4) \div (\pi R^2) = R^2/4 \quad . \quad . \quad . \quad . \quad (10)$$

APPENDIX II

1. Show that the effort P lb. required to pull a body weighing W lb. up a slope inclined at an angle θ to the horizontal is given by $P = W (\sin \theta + u \cos \theta)$, where u = the coefficient of friction between the body and the slope.

A lifeboat weighing 25 tons is pulled up a ramp inclined at 10° to the horizontal by a winch at a speed of 15 ft./min. If the efficiency of the winch is 75 per cent. and the coefficient of friction between the ramp and the lifeboat is 0·15, what horse-power will be required?

When launching the lifeboat what will be the free acceleration of the lifeboat down the ramp? (N.C.T.E.C.)

2. At noon, a ship A steaming at 20 m.p.h. north-west, is 12 miles due south of a ship B travelling at 25 m.p.h. due south. Determine (*a*) the velocity, in magnitude and direction, of the ship A relative to B, (*b*) the nearest distance the ships approach to one another, (*c*) the time when they are nearest to one another. (U.L.C.I.)

3. A, B and C are three points in the same plane in a rigid body having plane motion. AB = 5 ft., BC = 3 ft., AC = 4 ft. At a certain instant A is moving with a velocity of 20 ft./sec. in a direction A to B and at the same instant C is moving in the direction D to C, where D is a point on the line AB and AD = 2 ft. Find the instantaneous centre for the body. Hence determine (*a*) the angular velocity of the body, (*b*) the velocity of C, (*c*) the magnitude and direction of the velocity of B. (U.L.C.I.)

4. (*a*) In a velocity-time curve state what the slope of the curve at any point represents. What is indicated by the area between the velocity-time curve, the time axis, and the ordinates at the two given times?

(*b*) A train moving with uniform acceleration passes posts ½ mile apart as follows:

First post, 0 sec.; second post, 70 sec.; third post, 120 sec.

Determine the acceleration of the train and its speed when passing the first post. (N.C.T.E.C.)

5. A disc flywheel 5 ft. diameter and weighing 2500 lb. is keyed to a machine shaft. By the application of a constant torque the speed of the shaft is reduced from 240 r.p.m. to 180 r.p.m. in 8 sec. Determine (*a*) the magnitude of the torque, (*b*) the work done by the torque during the 8 sec., (*c*) the time taken by the flywheel in coming to rest from 180 r.p.m. if the torque remains constant. (U.L.C.I.)

6. Explain the function of a flywheel which is fitted to a reciprocating type engine.

The turning moment diagram for a petrol engine is drawn to the following scales: turning moment, 1 in. to 100 lb. ft.; crank angle, 1 in. to 25°. The fluctuation of energy between the crank positions for minimum and maximum speeds corresponds to an area of 0·95 in.^{2} of the turning moment diagram. Determine (*a*) the fluctuation of energy, (*b*) the weight of the flywheel required to prevent a fluctuation of speed greater than $\pm 0{\cdot}5\%$ from the mean speed of 1600 r.p.m. if the radius of gyration is 5 in. (U.L.C.I.)

7. A solid rectangular door of uniform thickness is 2·5 ft. wide and weighs 150 lb. It is hinged about a vertical axis on one of its longer sides. If a constant force of 10 lb. is applied normal to the plane of the door at a horizontal distance of 25 in. from the hinge, determine (*a*) the moment of inertia of the door about the vertical axis through its hinges, (*b*) the angular acceleration of the door, (*c*) the time taken for the door to open through 135°. Assume thickness of door is small and air resistance is negligible. (U.L.C.I.)

8. The flywheel of a shearing machine can be considered as a solid disc 5 ft. diameter and 6 in. thick. It is made of cast-iron of density 0·26 lb. per cu. in. and revolves at 120 r.p.m. Determine the uniform torque required to get up speed from rest in 2·5 min.

If the work done in shearing a plate is 15 ft.-tons, what will be the speed of the flywheel after shearing, assuming that no other energy is supplied? (U.L.C.I.)

9. In an experiment on a flywheel, a weight of 14 lb. was fixed to the end of a cord which was wound round the axle. The weight was allowed to fall vertically from a height of 4 ft. and the time required for the weight to reach the ground was 8·6 sec. On reaching the ground the weight became detached from the cord and the flywheel allowed to come to rest. The total number of revolutions made by the wheel from start to rest was 60. The effective diameter of the axle was 2 in. and the weight of the flywheel was 65 lb. Assume the frictional torque to be constant.

Calculate the radius of gyration (in inches) of the flywheel and the time taken, after detachment of the weight, for the flywheel to come to rest. (U.L.C.I.)

10. A swing bridge weighs 750 tons and swings about a pivot, the radius of gyration of the bridge about the axis of rotation being 62 ft. The bridge has to be turned through 90° in 70 sec., the first 30 sec. being a uniform angular acceleration period, the next 20 sec. a uniform angular velocity period, and the remaining 20 sec. a uniform angular retardation period. Determine (*a*) the maximum angular velocity of the bridge during a swing, (*b*) the angular acceleration, (*c*) the magnitude of the torque necessary to produce the angular acceleration. (U.L.C.I.)

11. Two trucks travelling in the same straight line collide

and remain locked together after impact. Truck A weighs 100 lb. and has a velocity of 10 ft. per sec. due East. Truck B weighs 150 lb. and has a velocity of 6 ft. per sec. due West. Determine the magnitude and direction of the velocity of the trucks after impact.

What is the total kinetic energy of the trucks (*a*) before impact, and (*b*) after impact? Account for any difference between (*a*) and (*b*). (N.C.T.E.C.)

12. A pulley weighing 40 lb. and having a radius of gyration of 16 in. has the following weights attached to it: 4 lb. at 20 in. radius; 7 lb. at 16 in. radius; 5 lb. at 14 in. radius; and 8 lb. at 10 in. radius. Determine the radius of gyration of the complete system.

If the outside diameter of the pulley is 24 in. and a force of 63 lb. acts in the periphery of the wheel, what will be the angular acceleration of the system? (N.C.T.E.C.)

13. A train, weight 250 tons, moving with a uniform velocity of 40 m.p.h. along a horizontal track, begins to climb up an inclined bank having a slope of 1 in 80. During the climb the engine exerts a constant tractive force of 5,000 lb. whilst the resistances to motion remain constant at 15 lb. per ton weight of the train. Determine how far the train will climb up the bank before coming to a standstill. Give your answer in feet. (N.C.T.E.C.)

14. Define Potential Energy, Kinetic Energy, and Work, stating the units in which they are measured.

A lorry weighing 15 tons is descending an incline 40 ft. long having a slope of 1 in 10. The speed of the lorry at the top of the incline is 20 m.p.h. Calculate the energy dissipated in bringing the lorry to rest at the bottom of the incline and the average force exerted by the brakes over the 40 ft. (N.C.T.E.C.)

15. A flywheel weighs 10 tons and has a radius of gyration of 3 ft. Find the time taken for the flywheel to reach a speed of 300 r.p.m. from rest if friction is neglected and the wheel is subjected to a constant torque of 500 lb. ft. Determine the number of revolutions the wheel would make in coming to rest from 300 r.p.m. under the action of a friction torque of 120 lb. ft. (N.C.T.E.C.)

16. Define moment of momentum.

A flywheel and shaft of total weight $\frac{1}{2}$ ton and radius of gyration 4 ft. are rotating in fixed bearings at 300 r.p.m. By means of a clutch on the shaft another similar system of flywheel and shaft of total weight 2 tons and radius gyration 2 ft. is suddenly connected to the first shaft. If the second flywheel and shaft are initially at rest find the new common speed of rotation and the loss in kinetic energy in the system due to the connection. (E.M.E.U.)

17. Two bodies weigh 5 lb. and 3 lb. respectively and hang on light strings 6 ft. long, side by side and just in contact. The larger weight is drawn aside, keeping the string taut, until its centre of gravity is raised through a vertical distance of $2\frac{1}{4}$ ft. It is then

released and on impact the two masses adhere. Calculate: (*a*) the velocity of 5 lb. weight just before impact; (*b*) the common velocity immediately after impact; (*c*) the loss of kinetic energy on impact; (*d*) the vertical height through which the centre of gravity of the combined system will rise; (*e*) tension in the string carrying the 5 lb. weight just before impact. (E.M.E.U.)

18. The total weight of a truck, including its four wheels, is 14 tons. Each wheel weighs 0·25 ton and may be assumed to be a solid disc 36 in. diameter. Calculate how much work is done on the truck on a level track in increasing its velocity uniformly from 15 m.p.h. to 30 m.p.h. Determine also the accelerating force if the truck moves 400 yd. during the operation. (E.M.E.U.)

19. Two masses of 20 lb. each are joined together by a string 4 ft. long, and are placed in contact with each other on a rough horizontal table. The coefficient of friction between the table and the masses is 0·2. A string from the mass nearer the edge of the table is carried over a pulley at the edge, and at the other end of this string is a mass of 5 lb. If this latter mass is released calculate: (*a*) the maximum velocity of the 5 lb. mass; (*b*) the velocity of the system just after the string between the 20 lb. masses has tautened; (*c*) the tension in this string during the motion after it has tautened.

Assume that the 5 lb. mass does not strike the floor and that the string is in line with the centres of the 20 lb. masses. (E.M.E.U.)

20. A flywheel weighs 4000 lb. and has a radius of gyration of 5 ft. It is acted upon from rest by a constant torque for 1½ minutes after which time it is rotating at 240 rev./min.

Calculate the value of the torque, the kinetic energy and the moment of momentum of the flywheel at 240 rev./min.

If the torque is now removed and the flywheel is uniformly retarded by a brake which brings it to rest in 100 revolutions, calculate the retarding torque exerted by the brake. (E.M.E.U.)

21. A cage weighing 2,000 lb. is lowered down a mine shaft which is 4,000 ft. deep. The cage moves with uniform acceleration for ten seconds, then with a constant velocity for fifty seconds, and finally retards at a uniform rate for fifteen seconds, at the end of which time it rests on the bottom of the shaft. Calculate the tensions in the hoisting cable during the three stages. (U.E.I.)

22. State the units in which the moment of inertia of a flywheel may be measured. A flywheel weighing 1 ton has a radius of gyration of 2 ft. Calculate: (*a*) the torque necessary to increase the speed of the flywheel from 200 rev./min. to 400 rev./min. in 20 seconds, assuming uniform acceleration, and (*b*) the average horsepower absorbed by the flywheel during this time. (U.E.I.)

23. The rim of a flywheel is 48 in. external diameter, 40 in. internal diameter and 6 in. wide. It is connected to the hub by a solid web. (*a*) If the specific weight of the material is 0·25 lb./in.^{3} determine the moment of inertia of the flywheel, stating

clearly the units employed. Work from first principles or prove any formula used. (The effect of the web and hub is to be disregarded.) (*b*) Calculate the energy in ft. lb. stored in the flywheel when rotating at 600 rev./min. (U.E.I.)

24. (*a*) Assuming the formula for centripetal acceleration, show that the height h of a conical pendulum is given by $h = g/\omega^2$, where ω is the angular velocity of the pendulum in radians per sec.

(*b*) A simple governor has arms 10 in. long pivoted on the axis of the governor spindle. Find the governor speed in r.p.m. when the arms are inclined to the spindle of the governor at an angle of 25°.

(*c*) What would be the angle of inclination of the arms to the governor spindle when the speed of the governor increases to 70 r.p.m.? (N.C.T.E.C.)

25. An engine weighing 200 tons travels around a curved track of 1,400 ft. radius. The centre of gravity of the train is 6 ft. above the rails which are 4 ft. 6 in. apart. Working from first principles, determine the super-elevation of the outer rail if side thrust is to be eliminated at the maximum allowable speed of 30 mp.h. You may assume the expression for centripetal acceleration.

Explain with the aid of a diagram the forces which are acting on the engine whilst travelling round the bend, giving their values. (N.C.T.E.C.)

26. Deduce an expression for the stress produced in a thin rotating rim due to centrifugal force.

A cylinder, which can be considered as a thin shell, is made of steel plate 0·375 in. thick and 6·0 ft. diameter, and is rotated about its horizontal axis. If the factor of safety is to be 8 and the ultimate stress of the steel used is 30·4 tons/in.^{2}, determine the maximum allowable rate of revolutions of the cylinder. Density of steel = 484 lb./ft.^{3} (U.L.C.I.)

27. Deduce an expression for the height of a conical pendulum or simple Watt governor when rotating at an angular speed of ω radians per second.

A bob is suspended from a point by a cord and rotated as a conical pendulum at 35 rev./min. and the radius of the path of the bob is 24 in. Determine the length of cord required. If the tension in the cord is 8 lb. calculate the weight of the bob. (U.L.C.I.)

28. A four-wheeled, two-axled railway vehicle weighs 30 tons and travels round a curve of 990 ft. radius at 40 m.p.h. Determine the side pressure of the wheel flanges on the rails.

The distance between the wheel tracks is 5 ft. and the centre of gravity of the vehicle is on the vertical through the centre of the wheelbase and 5·5 ft. above the ground. Calculate the speed (in miles per hour) at which the vehicle will be just on the point of overturning when rounding this curve. Also determine the necessary angle of banking of the track for the vehicle to travel round this curve at 60 m.p.h. and maintain equal pressure on each wheel. (U.L.C.I.)

29. Deduce an expression for the stress produced in a thin rotating rim due to centrifugal force.

A flywheel 4·5 ft. mean diameter weighs 222 lb. and can be considered as a thin rim 5 in. wide by 1 in. thick. Determine the hoop stress produced in the rim of the flywheel when rotating at 600 rev./min. (U.L.C.I.)

30. Show that the "height" of a conical pendulum is dependent only on the speed of rotation.

A roundabout has a number of carriages hung from a rotating roof by means of chains 12-ft. long. The upper ends of the chains are attached to the roof on a pitch circle of 10-ft. radius. Find the greatest speed at which the roundabout can operate if the angle which the chains make with the vertical is not to exceed 20 degrees. (U.E.I.)

31. Show that when a body moves in a circular path of radius R with a linear velocity V, the centripetal acceleration is equal to V^2/R.

The width between the wheels of a motor car is 5 ft. and when the car is loaded, its centre of gravity is $2\frac{1}{2}$ ft. above the ground in a plane midway between the wheels. If the maximum coefficient of friction between the tyres and the road is 0·5 show that the car will side slip and not overturn when cornering at speed on a level road.

Calculate the greatest speed at which a bend of 50 ft. mean radius can be negotiated without side slip. (U.E.I.)

32. Define the term *centripetal acceleration*.

The road surface of a hump-backed bridge has a radius of curvature of 27 ft. Calculate the greatest speed at which a car can cross the bridge without the wheels leaving the surface, if the centre of gravity of the car is 3 ft. above the ground. (U.E.I.)

33. A locomotive has its centre of gravity 4 ft. above the rails which are 5 ft. apart. At one section of the track the rails are at the same level and have a radius of curvature of 300 ft. At what speed would the locomotive overturn? At another section of the track where the radius of curvature is 1,800 yd. the outer rail is elevated in order that there shall be no lateral thrust on the rails at 60 m.p.h. (Show by means of a diagram what forces are acting on the loco.) Determine the difference in level of the two rails. (E.M.E.U.)

34. Obtain an expression for the acceleration of a body moving in a circular path with constant speed.

A 2 lb. mass is attached to a cord 2 ft. long and it is whirled round in a vertical circle at a speed of 120 rev./min.

Calculate the tension in the cord when the mass is: (*a*) at the top of the circle, (*b*) at the bottom of the circle.

Determine also the greatest speed at which the tension in the cord just disappears. Show the forces acting on the 2 lb. mass in its lowest position. (E.M.E.U.)

35. A body weighing 10 lb. oscillates in a horizontal straight line with simple harmonic motion. The amplitude of the oscillation is 36 in. and the maximum horizontal force acting on the

body is 60 lb. Determine (*a*) the frequency of the oscillations of the body per minute, (*b*) the velocity of the body at a point 18 in. from the mid-position, (*c*) the time taken for the body to travel 18 in. from one extreme end of the oscillation. (U.L.C.I.)

36. A body weighs 20 lb. and moves with simple harmonic motion. At a displacement of 12 in. from the centre of oscillation the velocity and acceleration of the body are 15 ft./sec. and 100 ft./sec.^{2} respectively.

Determine (*a*) the number of oscillations made per minute, (*b*) the amplitude of the motion, (*c*) the maximum velocity of the body, (*d*) the maximum force acting on the body. (U.L.C.I.)

37. Show that the periodic time for a complete oscillation of a body moving with simple harmonic motion is given by:

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}.$$

The reciprocating parts of an engine weigh 200 lb. and the stroke is 16 in. When the crank is 45° from the dead-centre position the force accelerating the reciprocating parts is 460 lb. Assuming that the motion of the reciprocating parts is simple harmonic, determine the speed of the crankshaft in revolutions per minute and the maximum velocity of the piston. (U.L.C.I.)

38. A weight W is hung on the end of a helical spring of stiffness S. Deduce an expression for the time of one complete vertical oscillation of the weight. Neglect the weight of the spring.

A 3 lb. weight is hung on the end of a helical spring and is set vibrating vertically. The weight makes 108 complete oscillations per minute. Determine the stiffness of the spring in pounds per inch. If the maximum extension of the spring during the vibration is 5 in. calculate the amplitude of the vibration. (U.L.C.I.)

39. Find an expression for the acceleration, at displacement x from its mid-position, of a body moving in a straight line with "Simple Harmonic Motion."

The piston of an engine has a stroke of 3 ft. and the reciprocating parts weigh 550 lb. If the crank shaft makes 250 r.p.m. find the force to overcome the inertia of the reciprocating parts at the ends of the stroke, and also the inertia force and the velocity of the reciprocating parts when the crank has turned 30 degrees from either dead centre. (U.E.I.)

40. Show that the period of vibration, T, of a simple pendulum of length L, is given by $T = 2\pi\sqrt{\frac{L}{g}}$

Hence or otherwise deduce the length, in inches, of the simple pendulum having a period of two seconds. (U.E.I.)

41. Define Simple Harmonic Motion.

A vertical helical spring having a stiffness of 4 lb./in. is clamped at its upper end and carries a weight of 6 lb. attached to the

lower end. The weight is displaced vertically a distance of 1 in. from the equilibrium position and released. Determine the period of the resultant vibration and the maximum velocity attained by the weight. (U.E.I.)

42. The piston of an engine is assumed to move with simple harmonic motion; the engine makes 300 r.p.m., the crank is 8 in. and the piston weighs 12 lb. Calculate the following: (*a*) accelerating force on the piston at the end of the stroke; (*b*) accelerating force on the piston at quarter stroke; (*c*) velocity of the piston at quarter stroke. (E.M.E.U.)

43. Define Simple Harmonic Motion and prove that the periodic time of vibration of a body moving with simple harmonic motion is given by:

$$t = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}.$$

A body weighing 180 lb. is oscillating in a horizontal direction with simple harmonic motion of amplitude 3 ft. and periodic time 12 sec.

Determine the maximum horizontal accelerating force.

(E.M.E.U.)

44. Derive an expression for the position of the centre of pressure of a vertical plane surface, immersed in a fluid, in terms of the first and second moments of the area.

A dock gate is 12 ft. broad and the water rises 12 ft. on one side and 6 ft. on the other above its lower edge. Determine the resultant force on the gate due to fluid pressure and the point at which this resultant acts. Density of water 62·4 lb./ft.^{3}

(U.L.C.I.)

45. State the Principle of Archimedes.

From one end of a balance is suspended a solid piece of iron and from the other is suspended a solid piece of copper, so that each hang totally immersed in a vessel containing oil weighing 40 lb./ft.^{3} and balance each other. The specific gravity of iron and copper is 7·8 and 8·8 respectively and the volume of the piece of iron is 10 in.^{3}

Find the volume of the piece of copper.

1 ft.^{3} of water weighs 62·5 lb. (E.M.E.U.)

46. A vertical sluice gate 3 ft. wide and 4 ft. deep has its upper edge 10 ft. below the free surface of the water. If the gate is held in position by one bolt at the top and one bolt at the bottom of the gate, calculate the force in each bolt.

Water weighs 62·5 lb./ft.^{3}

(E.M.E.U.)

47. Oil of density 59·3 lb./ft.^{3} flows along a pipe from a point A to a point B. At A the pipe diameter is 4 in. and the pressure is 20 lb./in.^{2} *abs* and at B, which is 6 ft. below A, the diameter is 6 in. and the pressure is 25 lb./in.^{2} *abs*. Determine (*a*) the velocity of the oil, in feet per second, at A, (*b*) the quantity of oil flowing

along the pipe in gallons per minute, (*c*) the total energy of the oil at A, in foot pounds per pound, assuming that the horizontal through B is the datum level. (1 ft.^{3} of oil = 6·24 gallons.) (U.L.C.I.)

48. The centre of a sharp-edged orifice 0·5 in. diameter, is situated 32 in. below the surface of the water in a tank with vertical sides and containing water to a depth of 36 in. Assuming a coefficient of velocity of 0·98, calculate how far the water in the jet issuing from the orifice will fall in moving horizontally a distance of 27 in. from the vena contracta. If the diameter of the jet at the vena contracta is 0·395 in. determine the coefficient of contraction. Also calculate the weight of water discharged in 10 min. assuming the head of water above the orifice remains constant. [1 ft.^{3} of water weighs 62·4 lb.] (U.L.C.I.)

49. State Bernoulli's theorem for the flow of a perfect incompressible fluid. Establish a formula for the theoretical discharge of water through a Venturi meter placed in a horizontal pipe-line.

A Venturi meter is 6 in. diameter at the entrance and 2 in. diameter at the throat. The difference in pressure between the entrance and the throat is equivalent to 4·5 in. of mercury. Determine the discharge, in gallons per minute, assuming no frictional losses. Density of water 62·4 lb./ft.^{3}, 1 gal. of water weighs 10 lb. and the specific gravity of mercury is 13·6. (U.L.C.I.)

50. Define the terms: (*a*) "limit of proportionality", and (*b*) "modulus of elasticity".

A pile used for the foundation of a building is made from a cast-iron cylinder 12 in. outside diameter, thickness 1 in. The inside of the cylinder is filled with concrete. If the load carried by the pile is 100 tons, find the stresses in the cast iron and the concrete.

[Young's modulus of elasticity for cast iron = 12×10^6 lb. per sq. in. Young's modulus of elasticity for concrete = 3×10^6 lb. per sq. in.] (N.C.T.E.C.)

51. Deduce expressions for the stresses produced in the longitudinal section (hoop stress) and the transverse section (axial stress) of a thin cylindrical shell subjected to an internal pressure.

A cylindrical shell 8 ft. internal diameter is constructed of mild steel plate $1\frac{1}{8}$ in. thick and having an ultimate tensile strength of 29·8 tons/in.^{2} It is subjected to an internal pressure of 180 lb./in.^{2} gauge and the efficiency of the longitudinal joint is 70%. Determine the factor of safety used in designing the shell. (U.L.C.I.)

52. The maximum safe compressive stress in a hardened steel punch is limited to 60 ton/in.^{2}, and the punch is used to pierce circular holes in mild steel plate $\frac{3}{4}$ in. thick. (*a*) If the ultimate shearing stress of the plate is 20 ton/in.^{2}, calculate the smallest diameter of hole that can be pierced. (*b*) If the effective length of the punch is 3 in., calculate the maximum strain energy stored

in the punch during the piercing operation. Assume the modulus of elasticity for the material of the punch to be 30×10^6 lb./in.^{2} (U.E.I.)

53. The sides of a step ladder weigh 10 lb. and 30 lb., are 8 ft. long and are joined at their centres by a rope 4 ft. long. The ladder rests on smooth ground.

Determine: (*a*) the tension in the rope, and (*b*) the magnitude and direction of the reaction at the hinge, when a man weighing 120 lb. stands on the heavier side at a distance of 2 ft. down the ladder measured from the hinge. (U.E.I.)

54. Prove the relationship $\frac{T}{J} = \frac{G\theta}{L}$ in connexion with simple torsion, where T is the torque which twists a solid, round shaft of length L through an angle θ. J is the polar second moment of area of the shaft section and G the modulus of rigidity of its material.

Determine the maximum allowable horse-power which can be transmitted by a 6-in. diameter, solid shaft running at 240 r.p.m., when the maximum permissible shear stress is 8,000 lb. per sq. in. The shaft carries a coupling having six bolts on a 10½-in. diameter pitch circle. Determine the diameter of the bolts if the maximum shear stress in the bolts must not exceed 6½ tons per sq. in. (N.C.T.E.C.)

55. Determine the external diameter of a hollow shaft required to transmit 10,000 h.p. at 50 r.p.m. if the maximum allowable shear stress is 3 tons/in., and the internal diameter is 0·5 of the external diameter.

What will be the angular deflexion of the shaft in degrees in a length of 10 ft. while the horse-power is being transmitted? [Modulus of rigidity = 12,000,000 lb./in.^{2}] (N.C.T.E.C.)

56. A circular shaft is subject to pure torsion. Establish a relationship between the angle of twist and the shear stress produced.

A shaft of 3·0 in. diameter when transmitting a torque twists through an angle of 3·1° over a length of 8 ft. and the maximum shear stress induced in the shaft is 10,000 lb./in.^{2} Determine the modulus of rigidity of the material of the shaft.

What will be the shear stress induced in the shaft at a radius of 1·0 in.? (U.L.C.I.)

57. A steel bar, 1·25 in. by 0·20 in. rectangular cross-section and 8 ft. long is laid symmetrically on two knife-edges at the same level 5 ft. apart, with the 1·25 in. dimension horizontal. A load of W lb. is hung at each end.

Sketch the bending moment diagram for the bar and show that the portion of the bar between the supports bends to form a circular arc. Determine the radius of this circular arc and the value of W if the maximum stress due to bending is not to exceed 8,000 lb./in.^{2} The modulus of elasticity of the material of the bar is 13,200 tons/in.^{2} (U.L.C.I.)

58. A horizontal beam ABC, 30 ft. long, is simply supported at the points A and B. The beam carries a uniformly distributed load of 1 ton/ft. between A and B and a concentrated load of W tons at C. AB = 20 ft.; BC = 10 ft. Determine the value of W if the bending moment at the centre of the beam is to be zero.

Using this value of W sketch (not necessarily to scale) the shear force and bending moment diagrams for the beam and determine the position and magnitude of the maximum bending moment on the beam. (U.L.C.I.)

59. The external diameter of a hollow shaft is equal to twice the internal diameter. When transmitting 1,000 h.p. at a speed of 150 rev./min. the angle of twist of the shaft must not exceed 1° over a length of 20 times the external diameter. Determine the required internal diameter if the modulus of rigidity of the material of the shaft is 5,500 tons/in.^{2}

Also calculate the maximum shear stress induced in the material of the shaft when transmitting the above power. (U.L.C.I.)

60. The floor of a room is supported on timber joists spaced 3 ft. apart, centre to centre, each being L ft. long between supports. The floor load, including the weight of the floor, is 240 lb./ft.^{2} and the joists are 10 in. deep by 4 in. thick. The 4 in. edge is horizontal.

Determine the value of L if the maximum stress produced in the beam due to bending is 1,000 lb./in.^{2}

What will be the maximum stress produced at a depth of 2 in. from the top face of the above beam? (U.L.C.I.)

61. A hollow shaft is 6 in. external and 3 in. internal diameter respectively and rotates at 180 rev./min. Determine (*a*) the maximum horse-power that can be transmitted by the shaft if the maximum shear stress in the material is not to exceed 8,000 lb./in.^{2}, (*b*) the angle of twist over a length of 20 ft. when the shaft is transmitting the maximum horse-power. The modulus of rigidity of the material is 12×10^6 lb./in.^{2} (U.L.C.I.)

62. A horizontal I-section girder, 20 ft. long, is simply supported at its ends and carries a uniformly distributed load (including its own weight) of 0·1 ton/ft. run and a load of W tons concentrated at the centre. The top and bottom flanges are 5 in. by 1 in., the total depth of the beam is 10 in. and the web is central and 0·5 in. thick. The beam is supported with the flanges horizontal. Determine the load W if the maximum stress in the material of the beam due to bending is not to exceed 10,000 lb./in.^{2} (U.L.C.I.)

63. A horizontal beam ABCD, 30 ft. long, is simply supported at the points B and C. The beam carries a uniformly distributed load of 1 ton/ft. over the whole length. AB = 4 ft.; CD = 6 ft.

Sketch (not necessarily to scale) the shear force and bending moment diagrams showing the values at the supports.

By calculation, determine the points between the supports of zero bending moment and the maximum value of the bending moment between the supports. (U.L.C.I.)

64. A horizontal lever 5 ft. long is made of solid steel bar of circular section. A vertical force of 75 lb. is applied at one end whilst the other end is held rigid. Determine the required minimum diameter of the bar if the ultimate tensile strength of the steel is 29·6 tons/in.^{2} and a factor of safety of 8 is used for bending stress produced.

Also determine the radius of curvature to which the loaded bar will bend at the section just outside the support. Modulus of elasticity of the steel is 13,000 tons/in.^{2} (U.L.C.I.)

65. The angle of twist of a solid shaft is not to exceed 1° over a length equal to 20 times the diameter of the shaft and the modulus of rigidity of the material is $11{\cdot}5 \times 10^6$ lb./in.^{2}

Determine (*a*) the maximum shear stress produced in the material of the shaft, (*b*) the required diameter of the shaft to transmit 200 h.p. at 240 rev./min. (U.L.C.I.)

66. A uniform shaft weighing 40 lb./ft. is supported in bearings 7 ft. apart and carries a flywheel weighing 490 lb. at a distance of 3 ft. from one bearing. Sketch the shear force diagram and determine the position and magnitude of the maximum bending moment in the shaft when it is stationary.

If the shaft rotates at 180 rev./min. and the centre of gravity of the flywheel is 0·3 in. from the axis of the shaft, determine the maximum forces on the two bearings. (U.L.C.I.)

67. A beam ABCD is 35 ft. long and is simply supported at B and C. The dimensions AB, BC and CD being 5, 20 and 10 ft. respectively. The beam is loaded throughout its length by a uniformly distributed load of 0·7 ton per foot and a concentrated load of 5 tons placed midway between the supports. Draw to scale the shearing force and bending moment diagram for the beam. (U.E.T.)

68. Establish the formula $\frac{p}{y} = \frac{M}{I}$ for a beam subjected to simple bending.

Determine the maximum uniformly distributed load a simply supported timber beam, 15 in. deep by 10 in. broad, can carry over a span of 10 ft. if the maximum permissible bending stress is 300 lb. in.^{2} (U.E.I.)

69. A hollow shaft having internal and external diameters of 2 and 3·5 in. respectively, transmits 2,000 h.p. when revolving at 2,000 r.p.m. The shaft is connected to another shaft by a simple flange coupling having eight bolts on a pitch circle of 7 in. diameter. Determine the minimum permissible diameter these bolts may possess, if the shear stress in the bolts is not to exceed 7,500 lb. in.^{2}

Calculate the maximum shear stress in the shaft under normal working conditions. (U.E.I.)

70. Explain carefully the meaning of the term *second moment of area* and state the units in which it may be measured. A steel pipe 4 in. external diameter and $\frac{1}{2}$ in. thick is used as a beam to span a distance of 8 ft. (*a*) Calculate the greatest central point load the pipe can carry if the longitudinal stress in the material is

not to exceed 8 ton in.^{2} (*b*) Compare the strength of this beam with that of a solid rod of the same cross-sectional area. (U.E.I.)

71. Prove the formula $q/r = T/J = G\theta/L$ for a circular shaft and state clearly the meaning of each of the symbols.

The propeller shaft of a car is in the form of a hollow tube 2 in. diameter and $\frac{1}{8}$ in. thick. Determine the maximum shear stress in the tube when the shaft transmits 70 horsepower at a speed of 4,000 rev./min. (U.E.I.)

72. A beam AB, 20 ft. long is freely supported at its ends A and B. It carries a uniformly spread load of 2 ton/ft. from A to the centre C, and also a concentrated load of 5 tons at D, 4 ft. from B. (*a*) Sketch the shear force and bending moment diagrams and insert on the diagrams the principal values. (*b*) State the position and value of the maximum bending moment. (U.E.I.)

73. A steel beam 12 ft. long rests symmetrically on supports 8 ft. apart, and carries equal loads, W, at each end. The beam is of I section having flanges 5 in. wide and $\frac{1}{2}$ in. thick, a web $\frac{3}{8}$ in. thick and an overall depth of 12 in. (*a*) If the maximum allowable longitudinal stress in the beam is 9 ton/in.^{2}, determine, in tons, the greatest permissible value of W. (*b*) For this value of W, calculate the height of the beam above the supports at the centre of the span. Assume E = 13,500 ton/in.^{2} (U.E.I.)

74. A hollow propeller shaft transmits 2,500 horsepower at a speed of 120 rev./min. The internal diameter of the shaft is two-thirds of the external diameter, and the maximum allowable shear stress in the material of the shaft is limited to 4 ton/in.^{2}

Determine (*a*) the external diameter of the shaft, and (*b*) the angle of twist over a length of 16 ft, if the shear modulus is 12×10^6 lb./in.^{2} (U.E.I.)

75. A beam ABCD is simply supported at B and C 10 ft. apart and the overhanging parts AB and CD are each 4 ft. long. The beam carries a uniformly distributed load of 2 tons per ft. run between A and C and there is a concentrated load of 9 tons at D. Draw to scale the shearing force and bending moment diagrams. (E.M.E.U.)

76. A rolled steel joist of I section has the following dimensions.

Top flange	6 in. wide × 1 in. deep,
Bottom flange	4 in. wide × 1 in. deep,
Web	$\frac{3}{4}$ in. wide × 6 in. deep.

It is used as a beam freely supported at each end, covering a span 12 ft. There is a concentrated load of 8 tons applied at each of two points, one point being 2 ft. and the other point being 10 ft. from the left-hand support.

Calculate the maximum bending stress in the beam and also determine the radius of curvature at the centre of the loaded beam.

Young's Modulus of Elasticity for material = 13,400 tons per sq. in. (E.M.E.U.)

77. The maximum shear stress in a hollow shaft 5 in. external diameter and 3 in. internal diameter is 4 tons per sq. in. when it is transmitting 320 h.p. Calculate the angle of twist on a 2 ft. length of the shaft and the speed of the shaft for the above conditions.

Modulus of Rigidity of shaft material $= 12 \times 10^6$ lb. per sq. in. (E.M.E.U.)

78. A beam 18 ft. long rests on two supports 10 ft. apart and overhangs the right-hand support by 3 ft. There is a concentrated load of 6 tons at the left-hand end and concentrated loads of 8 tons at 9 ft. and 12 ft. from the left-hand end. In addition each overhanging portion of the beam carries a uniformly distributed load of 1 ton per ft. run. Draw to scale the bending moment and shear force diagrams. (E.M.E.U.)

79. Prove the relationship which exists between the shear stress and the corresponding angle of twist of a round bar in torsion.

A solid steel shaft 2 in. diameter is subjected to torsion and the maximum shear stress induced is 4 tons per sq. in.

Calculate the angle of twist per foot length of the shaft if the modulus of rigidity of the shaft material is 5,200 tons per sq. in. (E.M.E.U.)

80. A beam of T section is used as a horizontal cantilever with the flange uppermost. The flange is 8 in. wide and 2 in. deep, and the web is 2 in. wide and 10 in. deep. If the cantilever carries a downward distributed load of 1 ton-ft. over its whole length, calculate the maximum length if the maximum stresses due to bending are not to exceed 5 tons per sq. in. in tension and 8 tons per sq. in. in compression. (E M.E.U.)

ANSWERS TO APPENDIX II (page 349)

1. 10·9 h.p.; 0·0259q.
2. (a) 41·65 m.p.h. 20° W. of N.; (b) 4·1 miles; (c) 16·25 min.
3. (a) 5 rad./sec.; (b) 18 ft./sec.; (c) 32·25 ft./sec. at 76° to BC.
4. 0·251 ft./sec.^{2}; 28·93 ft./sec.
5. (a) 190·5 Lb.-ft.; (b) 33,530 ft.-Lb.; (c) 24 sec.
6. (a) 41·56 ft.-Lb.; (b) 27·4 lb.
7. (a) 312·5 Lb.-ft.^{2}; (b) 2·146 rad. sec.^{2}; (c) 1·48 sec.
8. 35·87 Lb.-ft.; 9·895 r.p.m.
9. 7·47 in.; 59·8 sec.
10. (a) 2°/sec.; (b) $\frac{1}{15}$ degree/sec.^{2}; (c) 104·2 ton-ft.
11. 0·4 ft./sec. (Eastward); (a) 239·1 ft.-lb.; (b) 0·621 ft.-lb.
12. 15·5125 in.; 18·95 rad./sec.^{2}
13. 5,204 ft.
14. 583,300 ft.-lb.; 14,582 lb.
15. 393·3 sec.; 4,097 rev.

16. 150 r.p.m.; 137,400 ft.-lb.
17. (*a*) 12·037 ft./sec.; (*b*) 7·525 ft./sec.; (*c*) 4·22 ft.-lb.; (*d*) 0·8787 ft.; (*e*) 8·75 lb. **18.** 732,300 ft.-lb.; 610 lb.
19. (*a*) 3·21 ft./sec.; (*b*) 1·78 ft./sec.; (*c*) $2\frac{3}{4}$ lb.
20. 867·4 lb.-ft.; 981,000 ft.-lb.; 2,513,000 lb.-ft.^{2} rad./sec.; 1,562 lb.-ft.
21. 1,602 lb.; 2,000 lb.; 2,265 lb. **22.** 2,914 lb.-ft.; 16·65 h.p.
23. (*a*) 87·3 lb.-ft. sec.^{2}; (*b*) 172,300 ft.-lb. **24.** 62·24 r.p.m.; 44°.
25. 2·319 in. **26.** 909 r.p.m. **27.** 37·46 in.; 6·144 lb.
28. 82 m.p.h.; 13° 40′. **29.** 1,949 lb./in.^{2} **30.** 8·71 r.p.m.
31. 19·34 m.p.h. **32.** 21·19 m.p.h. **33.** 52·98 m.p.h.; 2·672 in.
34. 17·66 lb.; 21·66 lb.; 38·3 r.p.m.
35. (*a*) 76·63/min.; (*b*) 20·84 ft./sec.; (*c*) 0·1305 sec.
36. (*a*) 95·48/min.; (*b*) 1·803 ft.; (*c*) 18·03 ft./sec.; (*d*) 111·9 lb.
37. 119·7 r.p.m.; 8·353 ft./sec. **38.** 0·9928 lb./in.; 1·979 in.
39. 17,560 lb.; 15,210 lb.; 19·64 ft./sec. **40.** 39·156 in.
41. 0·3967 sec.; 16·04 in./sec.
42. 245·2 lb.; 122·6 lb.; 18·13 ft./sec.
43. 4·597 lb. **44.** 40,440 lb.; 4·665 ft. **45.** 8·774 cu. in.
46. Lower bolt 4,750 lb.; upper bolt 4,250 lb.
47. (*a*) 22·2 ft./sec.; (*b*) 725·6 gal./min.; (*c*) 62·21 ft.-Lb./lb.
48. 5·92 in.; 0·6241; 409·5 lb. **49.** 148·9 gal./min.
50. Cast iron 63·77 tons (1·845 ton/in.^{2}); concrete 36·23 tons (0·461 ton/in.^{2}).
51. 6·604. **52.** 1 in.; 0·3166 in.-tons.
53. 63·5 Lb.; on light side 80·8 Lb. inclined 52° to vertical.
54. 1,292 h.p.; 0·97 in. **55.** 21·68 in.; 0·3546°.
56. 11·82 × 10^6 lb./in.^{2}; 6,667 lb./in.^{2} **57.** 369·6 in.; 3·704 Lb.
58. 5 tons; 50 ton-ft. (at B). **59.** 3·754 in.; 5,391 Lb./in.^{2}
60. 7·86 ft.; 600 Lb./in.^{2} **61.** 908·4 h.p.; 3·056°.
62. 2·34 tons.
63. 0·883 ft. from B.; 1·883 ft. from C; 37·125 ton-ft.
64. 1·768 in.; 258·6 ft. **65.** 5,017 Lb./in.^{2}; 3·76 in.
66. R_A = 497·2 Lb.; R_B = 407·9 Lb. **68.** 750 Lb./ft. run.
69. 0·618 in.; 8,379 Lb./in.^{2} **70.** (*a*) 1·432 tons; (*b*) 2·362 times.
71. 1,696 Lb./in.^{2} **72.** 64 ton-ft., 8 ft. from A.
73. 12·9375 tons; 0·128 in. **74.** 9·76 in.; 1·683°.
76. 6·448 ton/in.^{2}; 9,317 in. **77.** 0·4106°; 105·4 r.p.m.
79. 0·530°. **80.** 9·247 ft.

ANSWERS TO EXAMPLES

Examples I (p. 20)

1. 350,000 ft.-lb. ; 800,000 ft.-lb.
2. 1,360,000 ft.-lb.
3. 5340 in.-lb.
4. 39,390 lb.-in.
5. 90·6 r.p.m.
6. 374·6° ; 0·482 second.

Examples II (p. 63)

1. Tension 21·68 lb. ; pressure 33·4 lb. ; 19·7° to vertical.
2. 0·1236.
3. 36°.
4. 15·3 lb. at hinge ; 8·25 lb. at free end.
5. 3950 lb. at A ; 2954 lb. at C.
6. 11·2 lb. cutting AD 2·1 in. from A, inclined 19·3° to DA.
7. 4·3 tons ; 3·46 tons ; 46·7° to horizontal.
8. 8·2 tons compression ; 4·39 tons tension ; 4 tons tension.
9. 8·78 tons tension ; 25·6 tons compression ; 21·2 tons tension.
10. 7·70 lb. ; 10·25 lb. ; 8·26 lb.
11. P = 3584 lb. ; 3985 lb. ; 1965 lb.
13. 17·7 tons left ; 11·3 tons right.
14. 21·6 lb. ; 134° clockwise.
15. 2·52 in. from outside of flange.
16. 3·67 in.
17. 1188 ft.-lb.
18. 75,600 ft.-lb.
19. 2510 ft.-lb.

Examples III (p. 93)

1. 103·4 lb.
2. 79·3 lb.
3. (*a*) 46·5 h.p. ; (*b*) 5·4 h.p.
4. 39·6 lb.
5. 125·7.
6. 9·8 tons.
7. 4·6 tons.
8. 18,095 lb.
9. 6750 r.p.m.
10. (*a*) 48 per cent. ; (*b*) 52 per cent. ; (*c*) 53 per cent.
11. $P = 0{\cdot}09 + 0{\cdot}072W$.
12. 26 lb. ; 4·96 ft./sec.^{2}.
13. $2l\omega \sin\theta$; $T/(2l \sin\theta)$.

Examples IV (p. 116)

1. 4·5, 14·6 and 11·4 ft. per sec.
3. 0·84 and 0·58 ft./(sec.)^2.
4. 77·3 ft. ; 2·9 sec.
5. 24 ft./sec. ; 71·05° forward ; 25·6 ft./sec. at −71·10° to horizontal ; 146 ft./sec.^{2} at 34° to vertical.
6. Radial $\omega^2 r$; tangential αr ; resultant $r\sqrt{\omega^4 + \alpha^2}$ inclined $\tan^{-1}(\alpha/\omega^2)$ to the radius.
7. 3·67 ft./sec. ; 1·31 radian/sec. ; 612 lb.-ft.

EXAMPLES V (p. 156)

1. 9802 lb.
2. 45·3 lb. per ton.
3. 23·44 ft./sec. ; 255,000 lb.
4. 1005 ft./sec.
5. 154 lb. ; 126 lb. ; 6·9 ft./sec.²
6. 99·2 lb.
7. (*a*) Front 45 lb., Rear 135 lb. ; (*b*) Front 78·54 lb., Rear 101·46 lb.
8. 350 lb.-ft.
9. 2·132 lb. (force) ft.-sec.² or 68·66 lb. (mass) ft.² ; 6·83 in.
10. 140·8 revs.
11. 7·71 in.
12. Acceleration $g(W_1 - W_2)/(W_1 + W_2 + W_3k^2/r^2)$;
$W_1(2W_2 - W_3k^2/r^2)/(W_1 + W_2 + W_3k^2/r^2)$;
$W_2(2W_1 + W_3k^2/r^2)/(W_1 + W_2 + W_3k^2/r^2)$.
13. 249·3 lb.-ft.
14. 106,900 ft.-lb. ; (*a*) 2138 ft. ; (*b*) 1069 ft.
15. 500,000 ft.-lb. ; 5220 lb.
16. 994·7 lb.-ft.
17. 7·246 ft./sec.
18. 7·5 ft.-lb. ; (*a*) 6·38 ft./sec. ; (*b*) 5·56 ft./sec.

EXAMPLES VI (p. 188)

1. 11·85 m.p.h.
2. 2672 ft.
3. 4·25 in.
4. 20 m.p.h.
5. 47° to horizontal.
6. 6·1°.
7. 15·83 tons ; 9·17 tons ; 5·4 in. ; 58·0 m.p.h.
8. 15 m.p.h. : 8·33 ft.
9. 19·45 m.p.h. 8·33 ft.
10. 1·835 ft. ; 43·8 and 35·78 r.p.m.
11. 132·7 r.p.m. ; 133·9 and 131·4 r.p.m.
12. 163·2 r.p.m. ; 150·3 r.p.m.
13. 4·0 lb. at 195·8° to first mass.
14. 43·2 lb. (near), 28·8 lb. (far).
15. 216 lb. (near), 144 lb. (far).
16. 127 lb., 246·8° from near, towards far crank.
17. 143·7 ft./sec.

EXAMPLES VII (p. 217)

1. 0·855, 1·56, 1·81 ft./sec.
2. 0·75 in.
3. 1654, 827, 1474 lb.
4. 153·3.
6. 0·342 second.
7. 1·103 second ; 67·3 ft./(sec.)².
8. 2·2 ft. ; 11·4 ft./sec. ; 226·8 ft./(sec.)² deceleration ; 2113 lb.
9. 1599 lb.-ft.
10. 3·56 in.
11. 125·5.
12. 31·23.
13. 1·728 ft. ; 25·96.
14. 38·33 and 35·68 ft./sec. ; 7·79 and 6·28 lb.
15. 29·82 ft./sec. ; 7·93W ; 19·34 ft./sec. ; 1·904W.
16. 1·034 second.

EXAMPLES VIII (p. 238)

1. 2621 lb. ; 3·714 ft.
2. 2353 lb. ; 3 ft. 4 in.
3. 0·6 ; 7·78.
4. 1·08 in.
5. 0·593.
6. 2° 45′.

Examples IX (p. 273)

1. 33,600 lb.
2. 153·2 gal.
3. 1361 lb. ; 1·94 h.p. ; 20·7 gal.
4. 20·8 amp. ; 4·16 pence.
5. 338 cu. ft.
6. 0·257 in.
7. 71·8 ft./sec. ; 1488 lb. per sq. ft.
8. 613 lb. per sq. ft.
9. 19,640 gal. per hr.
10. 18·95 ft.
11. 87·38 gal./min.
12. 3·21 in.
13. 2·68 in.
14. 632 ft. ; 20·19 ft./sec.
15. 423 lb. ; 868 lb.
16. 953 lb. ; 1620 lb.
17. 146·9 ft./sec. ; 72·2 per cent.
18. 1613 lb./min. ; 2·19.

Examples X (p. 289)

1. 0·00776 in. ; 0·00097.
2. 29,450,000 lb. per sq. in.
3. 15,300,000 lb. per sq. in.
4. 164 lb.
5. 0·376 in. ; 2·94.
6. 0·0318 in.
7. 0·367 ton ; 0·74 in.
8. 8640 lb. per sq. in.
9. 0·64 in.

Examples XI (p. 309)

4. 3820 lb. 40·9° to vertical, at A ; 1443 lb. vertical at B.
5. 48° ; 134·5 lb.
7. 4·06 ft.
8. 3·24 in.
9. 12 lb.
10. 53 ft.

Examples XII (p. 346)

1. 158 ton-ft. ; 20 tons ; 50 ton-ft. ; 14 tons.
2. 2650 ton-ft.
3. 8 ton-ft. ; 6 ft. from left end ; 9·75 ton-ft.
4. 11·75 ft. from A.
5. 32 and 40 ton-ft.
6. 4·6 ton-ft. ; 0·5 ton-ft.
7. 13 ton-ft.
8. 4·8 tons per sq. in.
9. 217·5 ton-in.
10. 15·625 tons ; 7·812 tons.
11. 1470 lb. per sq. in. ; 609·5 ft.
12. 3·33 in.
13. 13·1 in.
14. 12 ft.
15. 3·27 to 1.
16. 7 tons per sq. in.
17. 21,750 lb.-in.
18. 5091 lb. per sq. in.
19. 42,420 lb.-in. ; 3·05°.
20. 4·11 in. ; 2·23°.
21. 9906 lb. per sq. in.
22. 5304 lb. per sq. in. ; 1·012°.
23. 6·86 in. and 4·12 in.

MATHEMATICAL CONSTANTS

Constant	Number	Log	Reciprocal	Log reciprocal
π	3·1416	0·4972	0·3183	$\bar{1}$·5029
$\pi/4$	0·7854	$\bar{1}$·8951	1·2732	0·1049

PHYSICAL CONSTANTS

Constant	No.	Log.
Acceleration due to gravity (g) ft. per sec. per sec.	32·2	1·5079
Acceleration due to gravity (g) cm. per sec. per sec.	981	2·9917
Density of air at N.T.P. in lb. per cu. ft.	0·0807	$\bar{2}$·9069
Volume of air at N.T.P. in cu. ft. per lb.	12·39	1·0931
Density of water at 60° F. in lb. per cu. ft.	62·4	1·7952
Density of water at 60° F. in lb. per gallon	10·0	1·0000
Specific heat of air at constant volume	0·1691	$\bar{1}$·2283

CONVERSION FACTORS

To convert	Multiply by	Log	Reciprocal	Log reciprocal
Radians to degrees	57·2978	1·7581	0·0175	$\bar{2}$·2418
Metres to inches	39·3704	1·5952	0·0254	$\bar{2}$·4048
Miles per hour to ft. per sec.	1·4667	0·1668	0·6819	$\bar{1}$·8337
Pounds to kilograms	0·4536	$\bar{1}$·6567	2·2046	0·3433
Pounds (force) to dynes	445,000	5·6484	$2·247 \times 10^{-6}$	$\bar{6}$·3516
Horse-power to ft.-lb. per min.	33,000	4·5185	$3·031 \times 10^{-5}$	$\bar{5}$·4815
Horse-power to ft.-lb. per sec.	550	2·7404	0·001818	$\bar{3}$·2596
Horse-power to kilowatts	0·746	$\bar{1}$·8727	1·341	0·1273
B.Th.U. to lb. C.H.U.	0·5556	$\bar{1}$·7448	1·800	0·2553
B.Th.U. to gram-calories	252·0	2·4014	0·003968	$\bar{3}$·5986
Therms to B.Th.U.	100,000	5·00	0·00001	$\bar{5}$·00
Foot-pounds to B.Th.U.	0·001285	$\bar{3}$·1090	778·0	2·8910
H.p.-hours to B.Th.U.	2545	3·4057	0·0003929	$\bar{4}$·5943
kWh. to B.Th.U.	3,409	3·5326	0·0002934	$\bar{4}$·4675
kWh. to therms	0·03409	$\bar{2}$·5326	29·34	1·4675
Foot-pounds to joules	1·3562	0·1323	0·7374	$\bar{1}$·8677
Joules to ergs	10^7	7·00	10^{-7}	$\bar{7}$·00
Joules to gram-calories	0·2392	$\bar{1}$·3788	4·18	0·6212

TRIGONOMETRICAL FUNCTIONS

Angle.		Chord	Sine	Tangent	Co-tangent	Cosine			
Degrees.	Radians								
0°	0	0	0	0	∞	1	1·414	1·5708	90°
1	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	89
2	·0349	·035	·0348	·0349	28·6363	·9994	1·389	1·5359	88
3	·0524	·052	·0523	·0524	19·0811	·9986	1·377	1·5184	87
4	·0698	·070	·0698	·0699	14·3007	·9976	1·364	1·5010	86
5	·0873	·087	·0872	·0875	11·4301	·9962	1·351	1·4835	85
6	·1047	·105	·1045	·1051	9·5144	·9945	1·338	1·4661	84
7	·1222	·122	·1219	·1228	8·1443	·9925	1·325	1·4486	83
8	·1396	·140	·1392	·1405	7·1154	·9903	1·312	1·4312	82
9	·1571	·157	·1564	·1584	6·3138	·9877	1·299	1·4137	81
10	·1745	·174	·1736	·1763	5·6713	·9848	1·286	1·3963	80
11	·1920	·192	·1908	·1944	5·1446	·9816	1·272	1·3788	79
12	·2094	·209	·2079	·2126	4·7046	·9781	1·259	1·3614	78
13	·2269	·226	·2250	·2309	4·3315	·9744	1·245	1·3439	77
14	·2443	·244	·2419	·2493	4·0108	·9703	1·231	1·3265	76
15	·2618	·261	·2588	·2679	3·7321	·9659	1·218	1·3090	75
16	·2793	·278	·2756	·2867	3·4874	·9613	1·204	1·2915	74
17	·2967	·296	·2924	·3057	3·2709	·9563	1·190	1·2741	73
18	·3142	·313	·3090	·3249	3·0777	·9511	1·176	1·2566	72
19	·3316	·330	·3256	·3443	2·9042	·9455	1·161	1·2392	71
20	·3491	·347	·3420	·3640	2·7475	·9397	1·147	1·2217	70
21	·3665	·364	·3584	·3839	2·6051	·9336	1·133	1·2043	69
22	·3840	·382	·3746	·4040	2·4751	·9272	1·118	1·1868	68
23	·4014	·399	·3907	·4245	2·3559	·9205	1·104	1·1694	67
24	·4189	·416	·4067	·4452	2·2460	·9135	1·089	1·1519	66
25	·4363	·433	·4226	·4663	2·1445	·9063	1·075	1·1345	65
26	·4538	·450	·4384	·4877	2·0503	·8988	1·060	1·1170	64
27	·4712	·467	·4540	·5095	1·9626	·8910	1·045	1·0996	63
28	·4887	·484	·4695	·5317	1·8807	·8829	1·030	1·0821	62
29	·5061	·501	·4848	·5543	1·8040	·8746	1·015	1·0647	61
30	·5236	·518	·5000	·5774	1·7321	·8660	1·000	1·0472	60
31	·5411	·534	·5150	·6009	1·6643	·8572	·985	1·0297	59
32	·5585	·551	·5299	·6249	1·6003	·8480	·970	1·0123	58
33	·5760	·568	·5446	·6494	1·5399	·8387	·954	·9948	57
34	·5934	·585	·5592	·6745	1·4826	·8290	·939	·9774	56
35	·6109	·601	·5736	·7002	1·4281	·8192	·923	·9599	55
36	·6283	·618	·5878	·7265	1·3764	·8090	·908	·9425	54
37	·6458	·625	·6018	·7536	1·3270	·7986	·892	·9250	53
38	·6632	·651	·6157	·7813	1·2799	·7880	·877	·9076	52
39	·6807	·668	·6293	·8098	1·2349	·7771	·861	·8901	51
40	·6981	·684	·6428	·8391	1·1918	·7660	·845	·8727	50
41	·7156	·700	·6561	·8693	1·1504	·7547	·829	·8552	49
42	·7330	·717	·6691	·9004	1·1106	·7431	·813	·8378	48
43	7505	·733	·6820	·9325	1·0724	·7314	·797	·8203	47
44	·7679	·749	·6947	·9657	1·0355	·7193	·781	·8029	46
45°	·7854	·765	·7071	1·0000	1·0000	·7071	·765	·7854	45°
			Cosine	Co-tangent	Tangent	Sine	Chord	Radians	Degrees
								Angle.	

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	9	13	17	21	26	30	34	38
											4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	12	15	19	23	27	31	35
											4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	11	14	18	21	25	28	32
											3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	7	10	13	16	20	23	26	30
											3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	28
											3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	9	11	14	17	20	23	26
											3	5	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	14	16	19	22	24
											3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3	5	8	10	13	15	18	20	23
											2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
											2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
											2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	6	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	3312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	3	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
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78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
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83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
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85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
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ANTILOGS

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·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
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ANTILOGS

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·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4776	1	2	3	4	5	7	8	9	10
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·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
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·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
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·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
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·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

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